Generalised ‘Almost Impossible’ Chessboard Problem

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July 2020

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1 Introduction

The problem has the classic setup with an oddly math-obsessed warden offering their prisoners a chance at freedom behind a math puzzle. The aim is to come up with a strategy for the two prisoners to win the game and escape. The setup is as follows:

The warden has a standard 8x8 chessboard and places a coin on each square of the board. Every square has 1 coin and they can be placed with either heads or tails facing up. The warden then hides a key under one coin (we can assume each square on the chessboard has a compartment that can discreetly hide a key). The prisoners do not know beforehand which coins are heads and tails, or where the key is located.

As the game begins, prisoner 1 enters the room with the chessboard and is allowed to look at the orientation of the board and coins. The warden shows prisoner 1 where the key is located; Prisoner 1 must then flip exactly one coin and leave the room. Now, prisoner 2 enters the room and examines the chessboard position. Without altering the board's position, prisoner 2 must guess the location of the key to win.

The prisoners are allowed to discuss and come up with a strategy beforehand, however the warden is aware of the prisoners' strategy before setting up the coins and key. The prisoners are not allowed to communicate once the game begins. Is there a strategy the prisoners can use to win?

If you wish to try out the problem yourself, I would recommend pausing here before reading any further! You have been warned!

While these notes will cover the solution and some nice ways of looking at this problem, our main aim of will be to study a more generalised version of this problem. Specifically, given the same rules but instead arbitrary d-sided dice placed on a board with n-total squares (in the above case, $d = 2$ and $n = 64$), we will look at 1) the solvability of the problem - i.e. if a winning strategy exists (spoiler - it does not always exist) and 2) finding conditions on d, n to determine when the game is solvable.

2 The Original Problem

The aim of the problem is to find a way of reading the board such that each board state highlights a single square or 'key position' and the distance between any two key positions is exactly one coin flip. There are a fair few different models one might try in order to study the problem, and we will look at the two most popular ones.

2.1 Model 1

We will first look at a model that uses graph theory.

2x1 **Board**

On a 2x1 board, there are four possible board states. Let us represent each 'tails' as 0 and each 'heads' as 1.

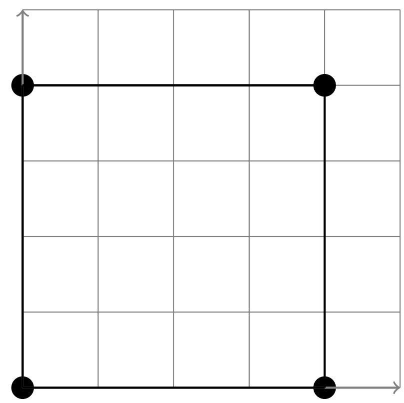
0	0

0	1

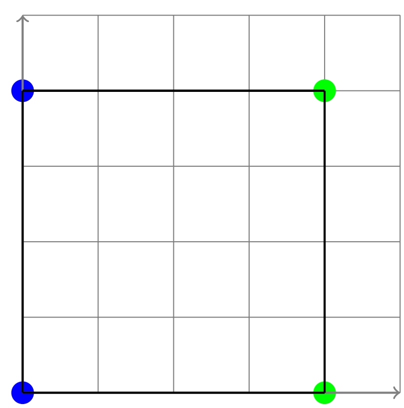
1	0

1	1

With this in mind, we can map each board state to the vertices of a square.



Notice that any two vertices of a square connected by an edge are separated by exactly one coin flip. Now, we know that there are two possible locations of the key. So assign each key location a color. Here, there are two colors. Solving the problem then becomes equivalent to coloring the square such that every vertex is adjacent to at least one vertex of every color. i.e If player 1 starts with a specific board state, or a specific vertex, they can reach either a vertex with one color - board state that represents the key at the first position - or a vertex with the other color - analogously, the second position. Player 2 need only note what the color of the vertex is given their board state. For this case, the coloring is simple:



3x1 Board

This model allows us to prove that the game is impossible in the 3x1 case. Here, there are 8 possible board states and hence we have the vertices of a cube.

0	0	0

0	0	1

0	1	0

0	1	1

1	0	0

1	0	1

1	1	0

1	1	1

Since there are 3 possible key positions, we must find a 3-coloring of the cube so that every vertex is adjacent to each of the three colors. This turns out to be impossible.

Showing some cases are impossible

We have a more general result.

Theorem 1. *For $n \neq 2^m$ for some m, there is no $n-coloring$ of a hypercube with 2^n vertices such that each vertex has one vertex of every color adjacent to it.*

Proof Sketch. Assume that such a coloring exists. Pick an arbitrary color, say red. Then every vertex has exactly one red neighbor. So the number of red neighbors is 2^n. Now given that every vertex has n-neighbours, we get that $2^n = n *$(the number of red vertices). This is not possible if n is not a power of 2.

We only provide a sketch here as we will prove stronger claims about this later.

2.2 Model 2 - The Solution

2x1 Board

Represent each tails as 0 and heads as 1. Let the board positions be indexed from 0. We say that the board sum s is the linear combination of board positioned scaled by their respective coin value. So for the following board

1	1

$s = 1(0) + 1(1) = 1$. This board sum represents the square associated to this board state, here it is the square at position 1 (which is the second square).

1	1

Now if the key is at position 0, then player 1 must flip the coin at position 1. If the key is at position 1, the coin at position 0 is flipped.

1	0
0	1
key	

0	1
0	1
	key

Player 2 can then calculate the board value and find where the key is.

Theorem 2. *Let s be the board sum on the $2x1$ board, k be the position of the key. Then if Player 1 flips coin $f = k - s \bmod 2$, the new board sum $s_2 = k$.*

Proof. By flipping the coin at f, we are adding f to the sum s. And by definition of f, $f + s = k$. Thus the resulting board sum is k. Note: even if the coin at f is 1, flipping the coin to 0 is the same as adding f. $s = x + f$ for some $x \in \{0, 1\} \implies s + f = x + 2f = x = s - f$. ∎

Original Problem

One might be tempted to solve the original case as follows: Let $s = c_0(0) + c_1(1) + c_2(2) + ... + c_{63}(63)$ be the linear combination of the board positions scaled by coin values c_i. Let k be the position of the key. Then the coin to be flipped is $f = k - s \bmod 63$. However, this fails. Consider the case:

0	1	0	⋯	0	0
0	1	2	⋯	62	63
		key			

Here $s = 1, k = 2$ so we need to add 1 to the board sum. However flipping the coin at 1 subtracts 1 from the total sum s, leaving the board value at 0.

0	0	0	⋯	0	0
0	1	2	⋯	62	63
		key			

In the 2x1 case, since we were working in addition modulo 2, subtracting 1 is the same as adding 1, but here, adding a position number and subtracting it are not equivalent. Adding 1 to s modulo 64 is equivalent to subtracting 63. If the coin at position 63 were a 1, then we would flip this and subtract 63 from the total. This is clearly impossible with the given case. Therefore this solution will fail.

To fix this, we will label the positions on the board in binary as follows:

000000	000001	000010	$\cdots$	111110	111111
(0)	(1)	(2)	$\cdots$	(62)	(63)

We will define addition to be digit-wise addition modulo 2 (This is equivalent to an XOR operation). We then get the result we want.

Theorem 3. *Let $s = c_0(b_0) \oplus c_1(b_1) ... \oplus c_{63}(b_{63})$ be the binary board sum, where c_i is the coin value at position i, b_i is the binary representation of i, and $\oplus$ is bitwise addition modulo 2 or XOR. Let k be the position of the key in binary. Then, under this definition of the board sum, flipping the coin at position $f = k \oplus s$ will result in a board sum of k.*

Proof. It is clear that $f = b_i$ for some $i \in \{0, 1, .., 63\}$. Now since $x \oplus x = 0$ for any position x on the board, we have that flipping the coin at f from 0 to 1 or vice versa is equivalent to adding f to the board sum. Then $s_2 = s \oplus f = s \oplus s \oplus k = k$. ∎

Note: We assumed above that $\oplus$ is associative and commutative, which is alright as the set of board positions in binary along with $\oplus$ form an abelian group. One can do the above steps without commutativity as well. It is simply more convenient for explanation purposes.

As an example game, consider the coin arrangement on the board as follows.

0	0	$\cdots$	0	0	0	$\cdots$	0	1
(000000) (0)	(000001) (1)	$\cdots$	101000 (40)	key	101010 (42)	$\cdots$	11110 (62)	11111 (63)

$s = 63_2 = 111111$ and $k = 41_2 = 101001$. So we need to add $f = 010110 = k \oplus s$ to s. So player 1 flips the coin at $010110 = 22_2$ to get a board:

0	0	$\cdots$	1	$\cdots$	0	$\cdots$	0	1
(000000) (0)	(000001) (1)	$\cdots$	010110 (22)		101001 (41) key	$\cdots$	11110 (62)	11111 (63)

Then, player 2 comes and finds that the board sum is now $41_2 = 101001$.

3 The General Chess Board Problem

3.1 Restating the Problem

In the original problem, the board consisted of 64 squares and coins with 2 sides. We will now look at the problem with the same rules, but a generalised setup. More specifically, the problem is:

The warden has a board with n total squares (The board need not be a square) and places a dice with d sides on each square of the board. Every square has 1 dice and they can be placed with either heads or tails facing up. The warden then hides a key under one coin (assume each square on the board has a compartment that can discretely hide a key). The prisoners do not know beforehand which coins are heads and tails, and where the key is located.

As the game begins, prisoner 1 enters the room with the board and is allowed to look at the orientation of the board and coins. The warden then shows prisoner 1 where the key is located. Prisoner 1 then flips exactly one dice to any of the other sides, and leaves the room. Now, prisoner 2 enters the room and examines the chessboard position. Without altering the boards position, prisoner 2 must guess the location of the key to win.

The prisoners are allowed to discuss and come up with a strategy beforehand, however the warden is aware of the prisoners' strategy before setting up the coins and key. The prisoners are not allowed to communicate once the game begins.

Questions:

1) Given some n, d, is the problem solvable?

2) Are there necessary and conditions on d and n that determine solvability?

When we use the term solvable, we will refer to solvability by method 2 of the original problem. Since this method also required a choice of labeling and operation (binary, instead of integers mod 64), we will work with the following definitions:

Notation 1. *Let there be a board of size n and arrangement of d sided dice on the board. Let there be a labeling $L = \{b_0, b_1, ..., b_n\}$ of the board positions with an operation $+$ on L. We denote $d_i(b_i) = b_i + b_i + ...b_i$ d_i-times. where d_i is the value of the dice on b_i (d_i will always be some natural number).*

Definition 1. *Let the board, dice, and labeling be as above. We define the **board sum** $s = d_0(b_0) + ... + d_n(b_n)$ where d_i is the value of the dice on b_i*

Definition 2. *Given $n, d \in \mathbb{N}$, we say a game with a board with n squares and d-sided dice is solvable iff there exists a labeling such that for any dice arrangement and key position k as in the rules of the game, there exists at least one dice flip such that the resulting board has sum k.*

Therefore, our question is: For what values of n, d is the game solvable?

3.2 Property of the labeling

As noted before, the choice of labeling is a significant component of the problem. We will require the following properties with our board-sum model:

1) The set of elements of board positions must have a closed operation $+$.

2) There must exist an identity element 0 under $+$. (This is because if the board sum is equal to the key position, we must be able to add 0 to the total.

3) Every position x must have an inverse under $+$. (This is in case the key is at 0 and the board sum is $y \neq 0$).

Thus we have our first fact.

Fact 1. *The labeling of a board with n squares under the operation forms a group.*

3.3 Properties of the Group

The criterion that the label is a group is not enough. As a quick example of this, we can look at the following 3x1 case from before:

0	0	1
0	1	2
	key	

Here, $n = 3, d = 2, k = 1$. There is only one group of order 3, that is $\mathbb{Z}/3$. Since the board sum $s = 2$, $f = k - s = 1 - 2 \pmod 3 = 2$. But we cannot add 2 or subtract 1 with the above board position. Flipping the coin at 2 subtracts 2 (or equivalently, adds 1) from the total, while flipping the coin at 1 adds 1 (or equivalently, subtracts 2) to the total.

0	0	0
0	1	2
	key	

0	1	1
0	1	2
	key	

This brings us to our main result.

Theorem 4. *Let $n, d \in \mathbb{N}$ be as before. The game is solvable if and only if $\exists$ a group of size n where the order of every element is at most d.*

Proof.

Forward Implication

Let the game be solvable. Therefore there exists a labeling G of size n of the board with property as in definition 2. Since every labeling forms a group, we will show that no element of G has order greater than d. We proceed by contradiction.

Let $x \in G$ be such that $\mathrm{ord}(x) = m \geq d + 1$. Recall that every element of G denotes a board position. Consider the following board arrangement:

0	0	$\cdots$	0	d	0	$\cdots$	0	$\cdots$	0	$\cdots$	0
g_0	g_1	$\cdots$	g_i	x	g_{i+2}	$\cdots$	dx	$\cdots$	$(d+1)x$	$\cdots$	g_n
									key		

Here, $s = dx \neq 0$. Note that $(d+1)x \neq dx$ since $x \neq 0$. Now given that the game is solvable, there exists $y \in G$ and $0 \leq d_y \leq d \in \mathbb{N}$ such that $dx + d_y y = k = (d+1)x$ and $d_y y \neq dx$ if $x = y$ (Since player 1 must flip some dice and cannot leave the board state unaltered). Thus $d_y y = x = -(m-1)x \neq 0 \pmod{n}$. So we must either add x or subtract $(m-1)x$. But neither of these is possible. Since the dice at x is at d, we cannot add x to get $(d+1)x$ with this dice as every dice only has d-sides (and $(d+1)x \neq 0$. We cannot subtract $(m-1)x$ because $(m-1)x \neq x$ and so has a dice with 0 value. So we cannot subtract $(m-1)x$ by changing this dice value.

We then get that the game is not solvable, which is a contradiction. Therefore, for any $x \in G$, the order of x is at most d.

Backward Implication

Let G be a group of order n, with every element of at most order d. Pick any one-to-one mapping of the elements of G onto the squares of the board. Consider an arbitrary board arrangement.

d_0	d_1	$\cdots$	d_i	d_k	d_{i+2}	$\cdots$	d_n
g_0	g_1	$\cdots$	g_i	k key	g_{i+2}	$\cdots$	g_n

Now we have $s = \sum_{i=0}^{n} d_i g_i \in G$. Let $f = s - k \in G$. Clearly $s + f = k$. Now consider the dice on f, d_f. Case 1: $d_f < d$. Then flipping the dice to $(d_f + 1)$ will make the board sum $s + f = k$. Case 2: if $d_f = d$. Since $\text{ord}(f) \leq d$, either $\text{ord}(f) = d$ or $\text{ord}(f) < d$. In the first case, set $d_f = 0$ and in the latter case, set d_f to $d + 1 - \text{ord}\ (f)$ (If $f = 0$, then just change the dice to any other number). ■

Examples

1) For the $n = 3, d = 2$ case from before, we see that the only group of order 3 is $\mathbb{Z}/3$, in which every element has order $3 > d = 2$.

2) For the $n = 64, d = 2$ case (the original problem), we see that $\mathbb{Z}/64$ fails as it has elements of order greater than 2, but our binary labeling, i.e. $(\mathbb{Z}/2)^6$ worked since every element has order at most 2.

3) For $d = 2$, The only groups that will work are $(\mathbb{Z}/2)^m$. And so we see that theorem 1 and the original problem are special cases of this.

3.4 Programmability and Relations between d and n

While the above result explains the structure of the underlying problem well, it is not a good condition if one were to try to write a program to check whether an arbitrary case, given n, d is solvable. Therefore, we will now try to classify what cases are solvable by examining conditions on d, n and studying when such groups exist.

Case 1: $d \geq n$

This case is straightforward. For $d \geq n$, consider the Group $\mathbb{Z}/n$. The order of every element is at most n and so the game is solvable. There is, in some sense, more than enough information that each dice can provide.

Case 2: $d < n$

This case is the more interesting one. Clearly not every case here is solvable as noted before. We have then the following results.

Theorem 5. *Let the prime factorization of n be $n = \prod_{i=0}^{m} p_i^{k_i}$. If $d < p_m$, then the game is unsolvable.*

Proof. This follows from Cauchy's theorem, that for any finite group G of order n, there exists an element of order p_i for every $i \in \{0, ..., m\}$. ∎

Theorem 6. *Let the prime factorization of n be as in theorem 5. If $n > d \geq \prod_{i=0}^{m} p_i$, then the game is solvable.*

Proof. Consider the group $(\mathbb{Z}/p_0)^{k_0} \times (\mathbb{Z}/p_1)^{k_1} \times ... \times (\mathbb{Z}/p_m)^{k_m}$. The order of every element in this group is at most $\prod_{i=0}^{m} p_i$ ∎

Theorem 7. *Let the prime factorization of n be as in theorem 4. If $p_m < d < \prod_{i=0}^{m} p_i$, then there is no finite abelian group that satisfies theorem 4.*

Proof. This follows from the classification of finite abelian groups, which states that any finite abelian group is a product of cyclic groups of prime-power order. Therefore in any finite abelian group of order n, there will exist at least one element of order $p_0 p_1 ... p_m$. ∎

Final Question

Given our results for case 2, it is only left to check for what values of d between p_m and $\prod_{i=0}^{m} p_i$ the game is solvable. There are examples of both solvable and unsolvable game with these restrictions:

Solvable: Let $n = 6, d = 5$. Here the non-abelian group of permutations of three elements, S_3, works as a labeling and one can check that this is indeed a valid labeling of the board.

Unsolvable: Let $n = 28, d = 8$. There is no group that satisfies theorem 4 and therefore this case is unsolvable. In fact, for every $d < 14$ there are no solutions.

We can find that certain groups are helpful to look at. If we look at the dihedran group, for example, we get the following result:

Fact 2. *If $n = 2p_1...p_m$ where $p_i \neq 2$, then for any $\prod_{i=0}^{m} p_i \leq d < n$, the game is solvable by taking the Dihedral group $D_{n/2}$*

Similarly, looking at the permutation groups S_n we see:

Fact 3. *If $n = m!$ for some $m \in \mathbb{N}$, then for $n \leq d$ the game is solvable by taking the group S_n*

Finding a general criterion with the remaining cases would then mean classifying for arbitrary values of n the lowest possible maximal-element-order of every non-abelian group of order n. What the criterion is is not clear yet.