Corollary 1. *If $\xi''' = \nu''' = 0$, then the only models that admit backward models are when ϕ is linear.*

Proof. By the proof of the main theorem, we have that

$$\frac{\partial^2 \pi}{\partial x \partial y} \cdot \frac{\partial}{\partial x}\left(\frac{\partial^2 \pi}{\partial x^2}\right) = \frac{\partial^2 \pi}{\partial x^2} \cdot \frac{\partial}{\partial x}\left(\frac{\partial^2 \pi}{\partial x \partial y}\right)$$

which, when substituting the equations for each term obtained in the proof of the theorem, gives us that

$$(\nu''\phi') \cdot \frac{\partial}{\partial x}(\nu''(\phi')^2 - \nu'\phi'' + \xi'') = (\nu''(\phi')^2 - \nu'\phi'' + \xi'') \cdot \frac{\partial}{\partial x}(-\nu''\phi')$$

$$\implies -2(\nu''\phi')^2\phi'' + \nu''\nu'\phi'''\phi' - \xi'''\nu''\phi' = -\nu'\nu'''\phi''(\phi')^2 + \xi''\nu'''(\phi')^2 + \nu''\nu'(\phi'')^2 - \nu''\phi''\xi'' \tag{1}$$

By assumption, since $\xi''' = \nu''' = 0$, we know that $\nu'' = C$ for every $y - \phi(x)$. Furthermore, since $\nu = \log P_{N_Y}$, we cannot have $C = 0$ by chain rule. Putting this together with (1), we get

$$\nu' \cdot (\phi'\phi''' - (\phi'')^2) = 2C(\phi')^2\phi'' - \phi''\xi''$$

Since $C \neq 0$, we know that ν' is linear and with nonzero slope, thus there must exist an α such that $\nu'(\alpha = 0$. Thus, restricting to the set on which $\nu' = 0$, we get

$$\phi'' \cdot (2C(\phi')^2 - \xi'') = 0$$

Assume, for the sake of contradiction, that there is an x for which $\phi''(x) \neq 0$. Then we have $(\phi'(x))^2 = \frac{\xi''}{2C}$. Since we know that $\xi''' = 0$, similar to ν, we get that $\xi'' = D \neq 0$. So then $(\phi'(x))^2 = \frac{D}{2C} \implies \phi'(x) = \sqrt{\frac{D}{2C}}$ and thus ϕ' is a constant function. This is true for every x on which $\phi''(x) \neq 0$. But the set $S = \{x : \phi''(x) \neq 0\}$ is an open set. This we have an open set S on which $\phi'(x)$ is a constant function $\forall x \in S$. Thus, on the open set S, we have $\phi''(x) = 0 \forall x \in S$. This contradicts the definition of S. Thus there is no such x and $\phi''(x) = 0$ $\forall x$ and so ϕ is linear. ∎