A Perturbation Bound on the Subspace Estimator from Canonical Projections

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Outline

1. Problem Setup - Subspace Estimation
2. Motivation - Missing Data
3. Previous Work: Noiseless case
4. This Paper - Noisy Data and Estimation Bound
5. Applications
6. Conclusions

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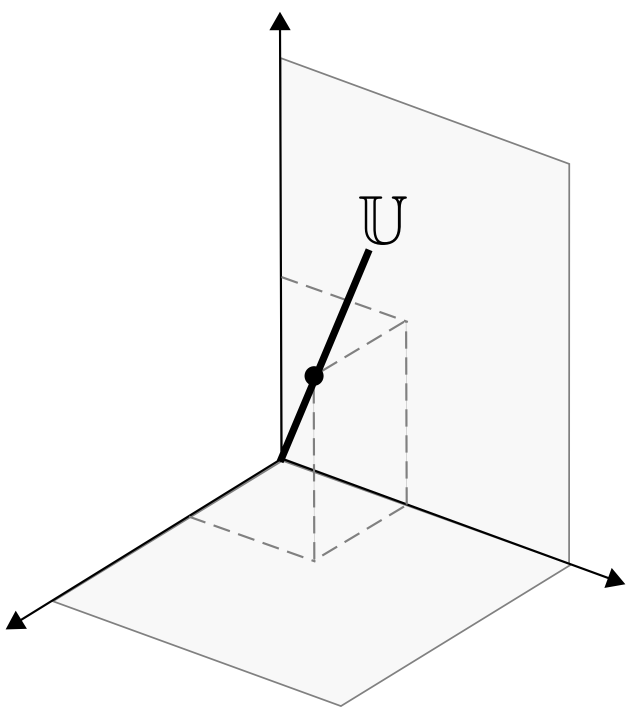
$\mathbb{U} :=$ $r-$ dimensional subspace of $\mathbb{R}^d, r < d$



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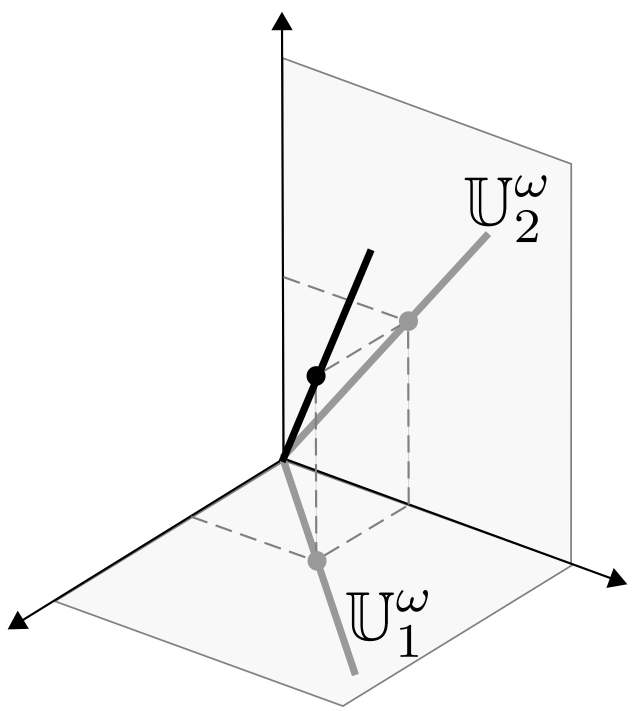

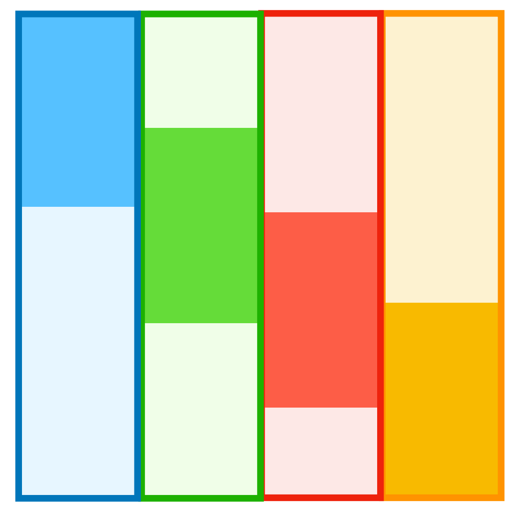

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$\mathbb{U} :=$ $r-$ dimensional subspace of $\mathbb{R}^d, r < d$

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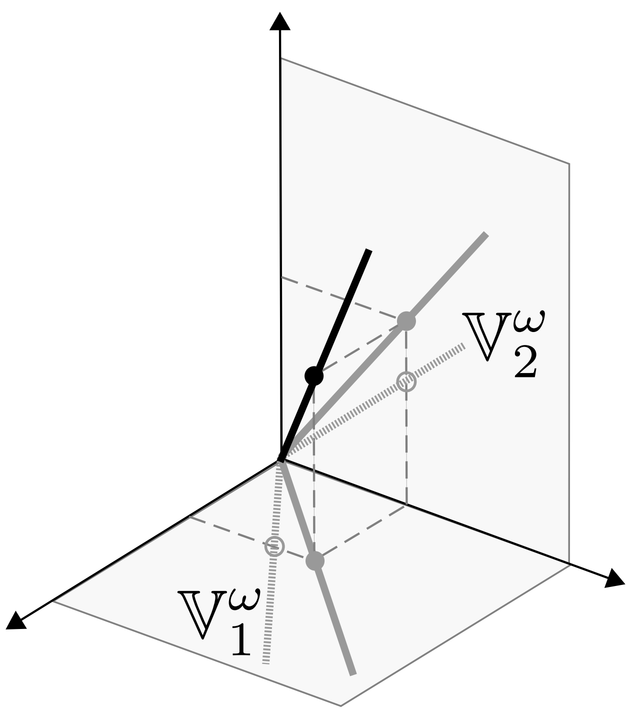

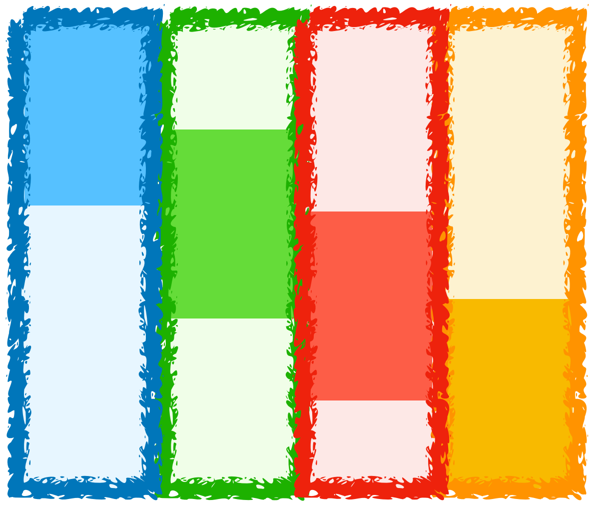

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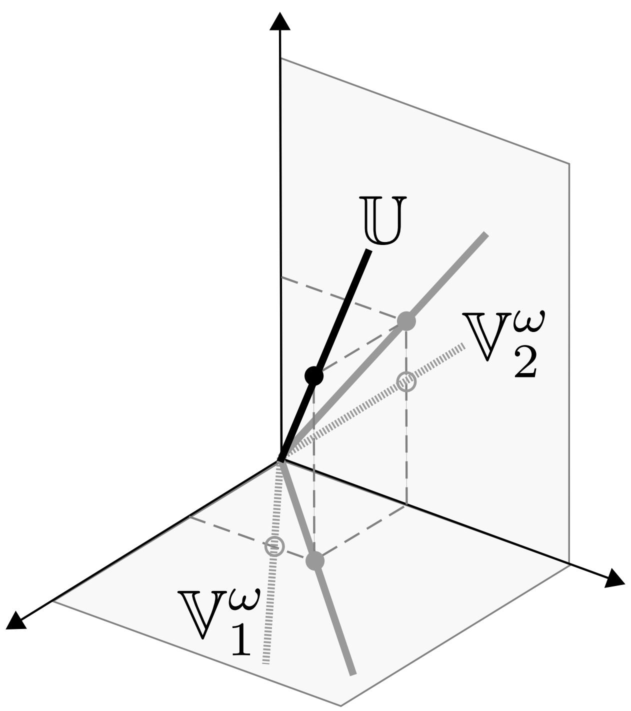

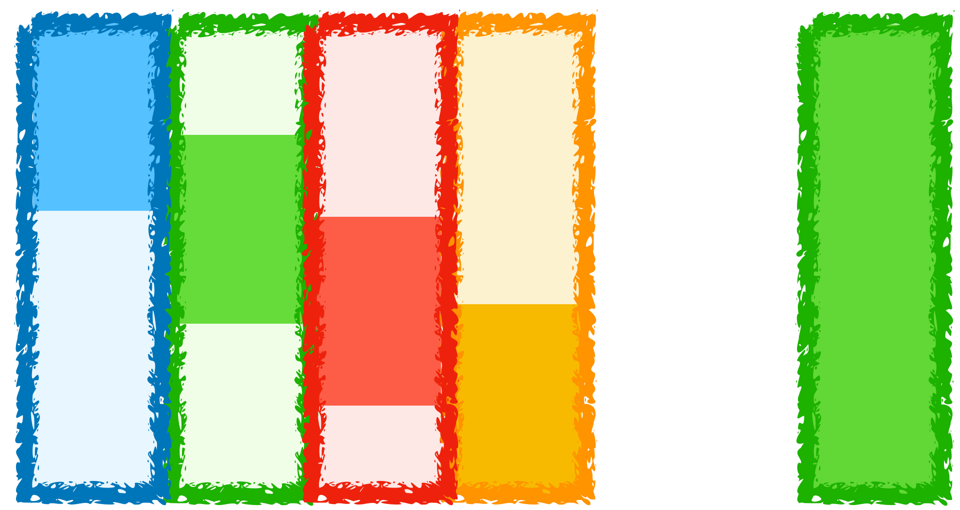

Goal: To estimate $\mathbb{U}$ from the $\{\mathbb{V}_i^\omega\}'$ s and bound the error

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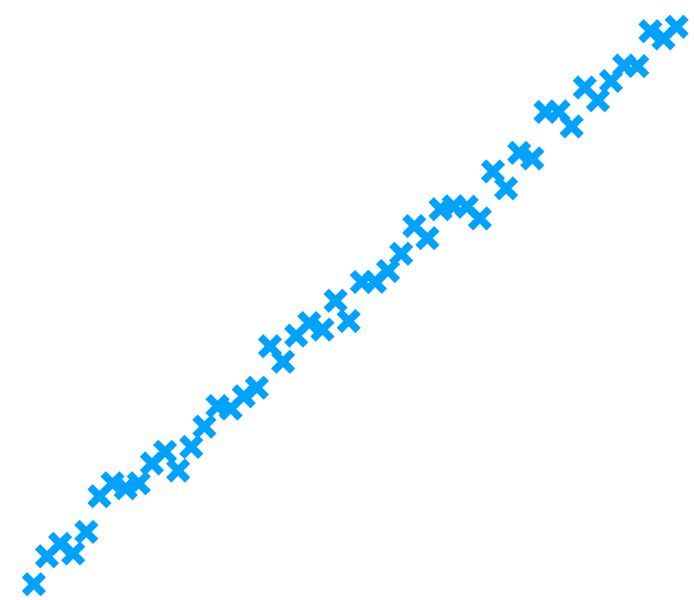
Motivation

The real world has lots of data

Motivation

Our main tool for modelling data is linear algebra

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 & 6 \\ 1 & 2 & 1 & 3 & 2 & 6 \\ 1 & 2 & 1 & 3 & 2 & 6 \\ 2 & 4 & 2 & 6 & 4 & 12 \\ 2 & 4 & 2 & 6 & 4 & 12 \end{bmatrix}$$



Since data is often best modelled by subspaces

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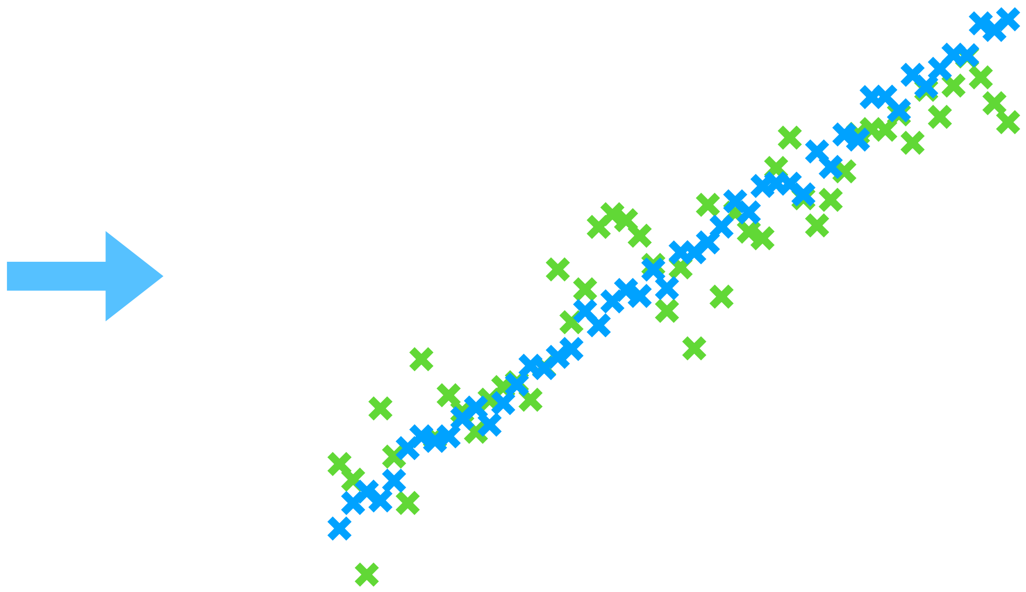
1.1	1.9	0.8	3.1	2.2	5.8
0.9	2.2	1.2	3.2	2.1	6.2
1.2	1.8	1.1	2.9	1.8	5.9
2.1	3.9	2.2	5.8	4.1	12.1
1.9	4.1	2.1	5.9	4.2	12.2



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But data is often noisy

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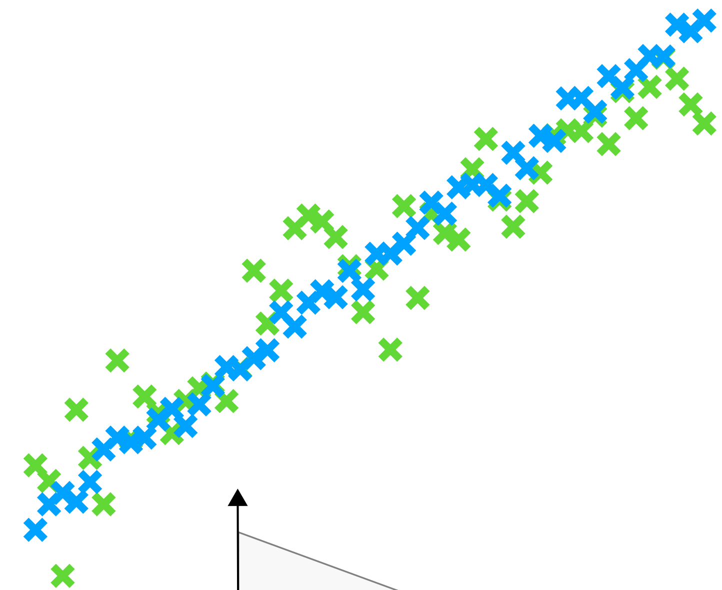

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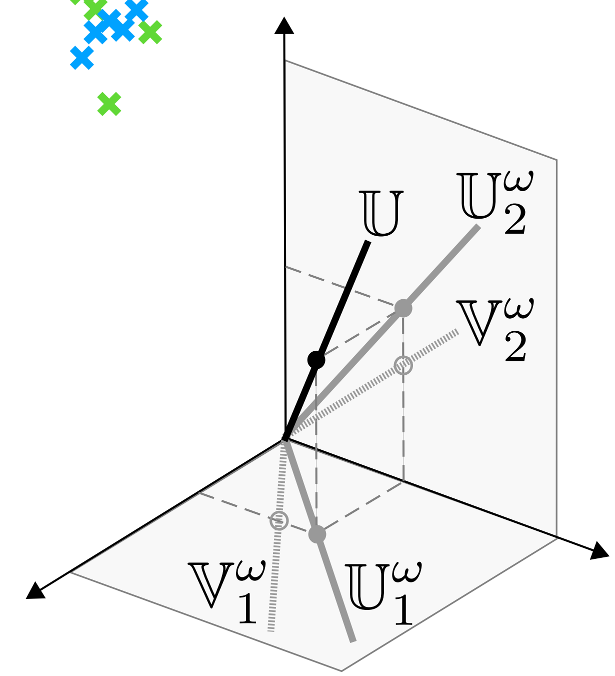

Since data is often best modelled by subspaces

But data is often noisy and missing

Motivation

$$\begin{bmatrix} 1.1 & 1.9 & 0.8 & \cdot & 2.2 & \cdot \\ 0.9 & 2.2 & \cdot & \cdot & 2.1 & 6.2 \\ 1.2 & \cdot & 1.1 & \cdot & 1.8 & 5.9 \\ \cdot & \cdot & 2.2 & 5.8 & \cdot & 12.1 \\ \cdot & 4.1 & 2.1 & 5.9 & 4.2 & \cdot \end{bmatrix} \longrightarrow \begin{bmatrix} 1.1 & 1.9 & 0.8 & 3.1 & 2.2 & 5.8 \\ 0.9 & 2.2 & 1.2 & 3.2 & 2.1 & 6.2 \\ 1.2 & 1.8 & 1.1 & 2.9 & 1.8 & 5.9 \\ 2.1 & 3.9 & 2.2 & 5.8 & 4.1 & 12.1 \\ 1.9 & 4.1 & 2.1 & 5.9 & 4.2 & 12.2 \end{bmatrix}$$

The problem of completing matrices is
(Low Rank) Matrix Completion

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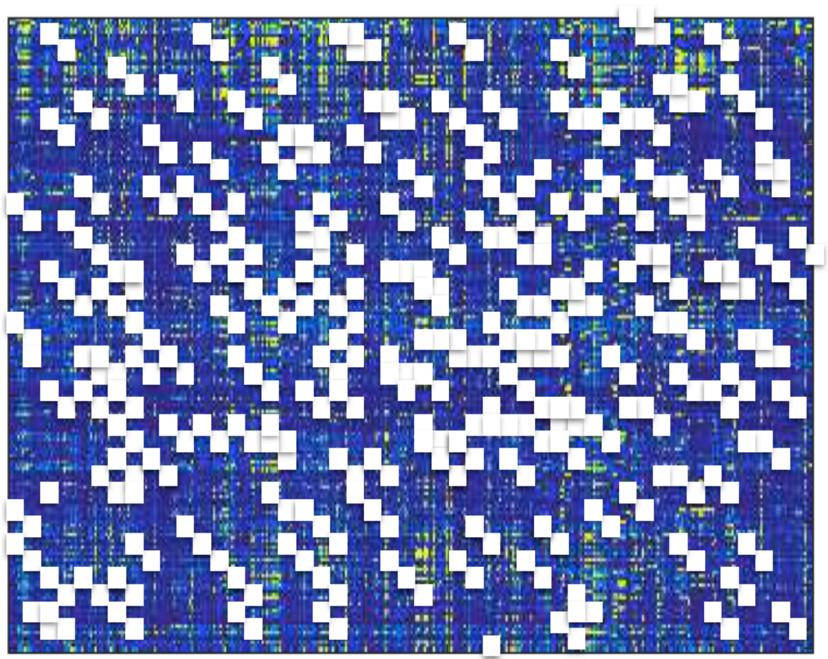
$$\begin{bmatrix} 1.1 & 1.9 & 0.8 & \cdot & 2.2 & \cdot \\ 0.9 & 2.2 & \cdot & \cdot & 2.1 & 6.2 \\ 1.2 & \cdot & 1.1 & \cdot & 1.8 & 5.9 \\ \cdot & \cdot & 2.2 & 5.8 & \cdot & 12.1 \\ \cdot & 4.1 & 2.1 & 5.9 & 4.2 & \cdot \end{bmatrix} \longrightarrow \begin{bmatrix} 1.1 & 1.9 & 0.8 & 3.1 & 2.2 & 5.8 \\ 0.9 & 2.2 & 1.2 & 3.2 & 2.1 & 6.2 \\ 1.2 & 1.8 & 1.1 & 2.9 & 1.8 & 5.9 \\ 2.1 & 3.9 & 2.2 & 5.8 & 4.1 & 12.1 \\ 1.9 & 4.1 & 2.1 & 5.9 & 4.2 & 12.2 \end{bmatrix}$$

The problem of completing matrices is (Low Rank) Matrix Completion

Targets (proteins)

Drug Compounds



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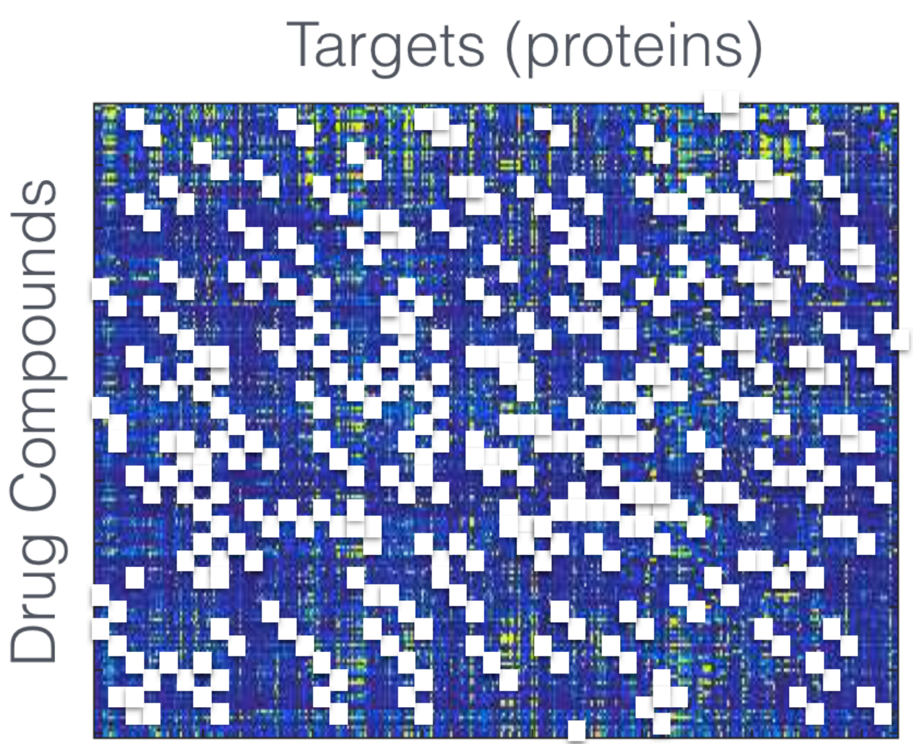

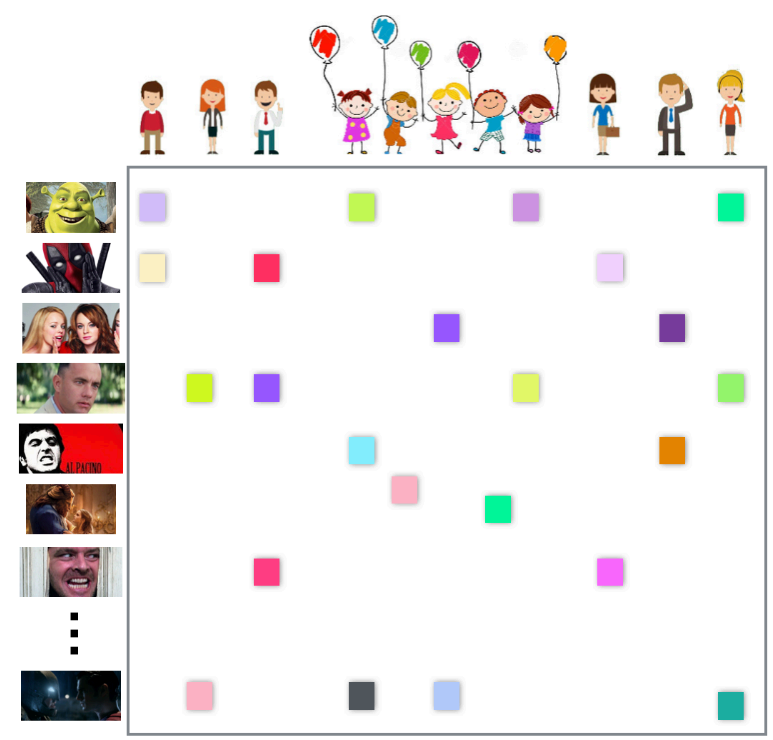

Motivation

Subspace Reconstruction sheds light on these problems

More to come later!

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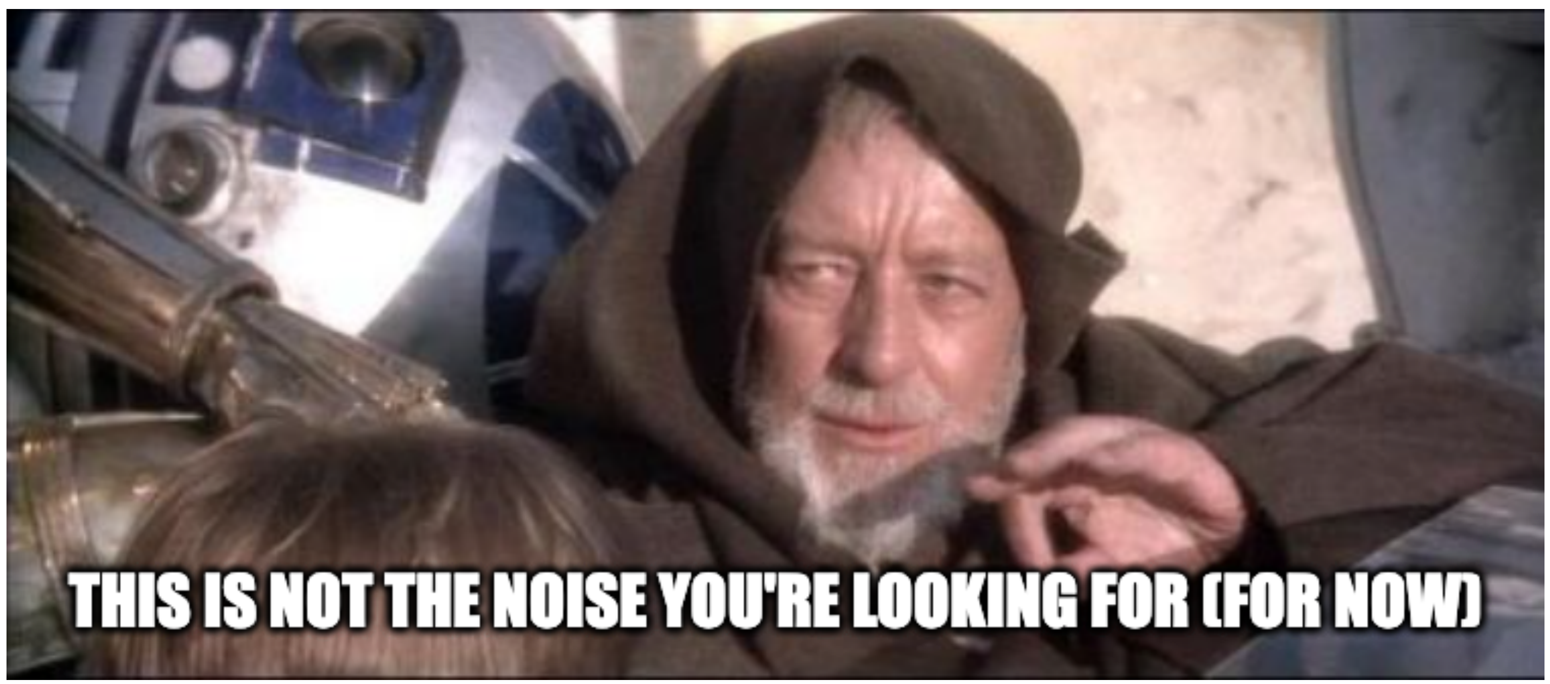
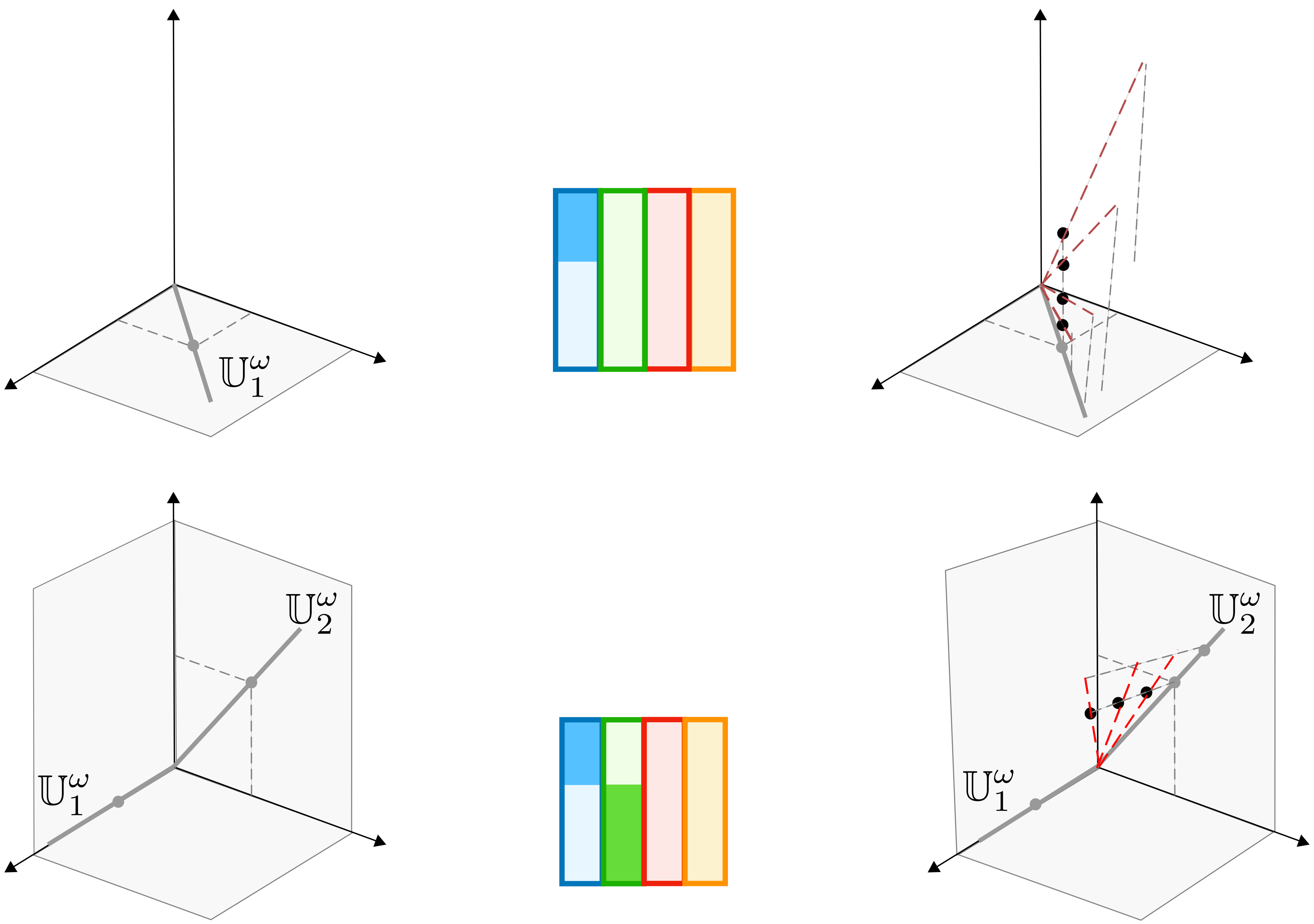
Previous Work - Noiseless Case

Let's forget about noisy data for a second.



Previous Work - Noiseless Case

Many projections don't have unique subspace estimations



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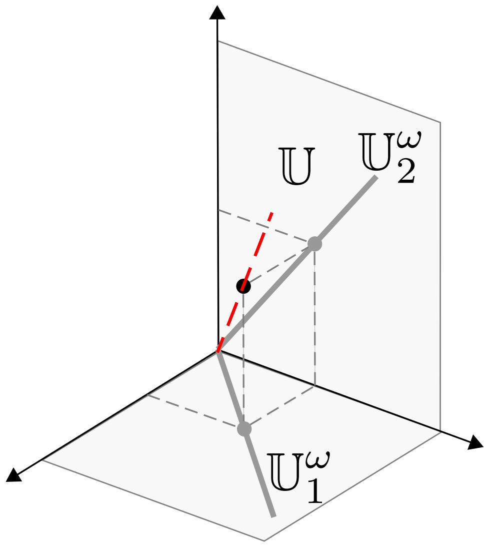
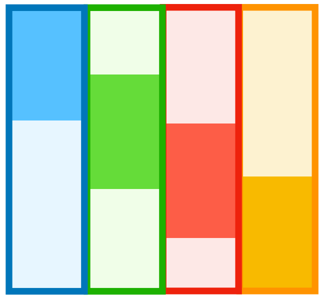
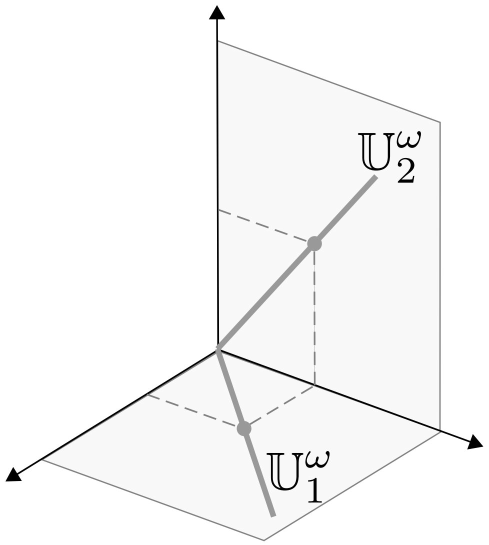
Many projections don't have unique subspace estimations, but many do.







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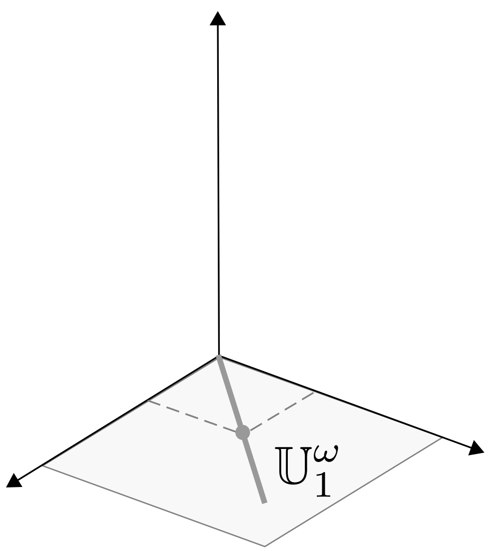

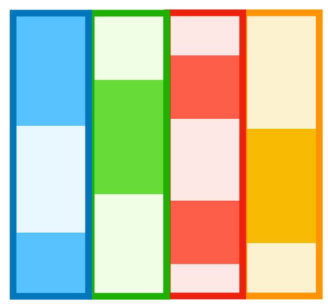

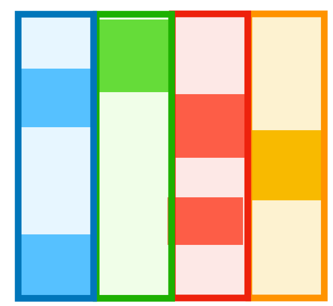

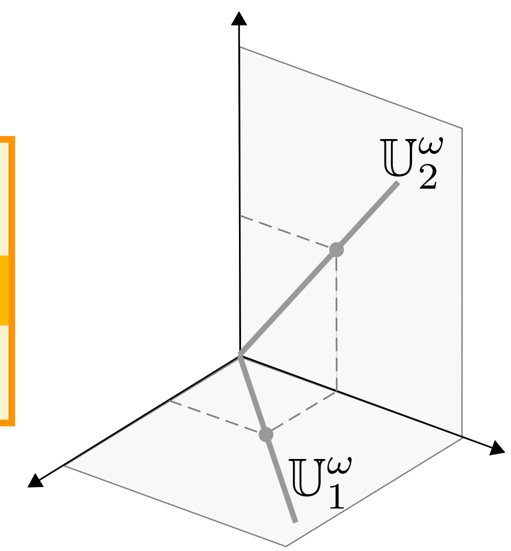

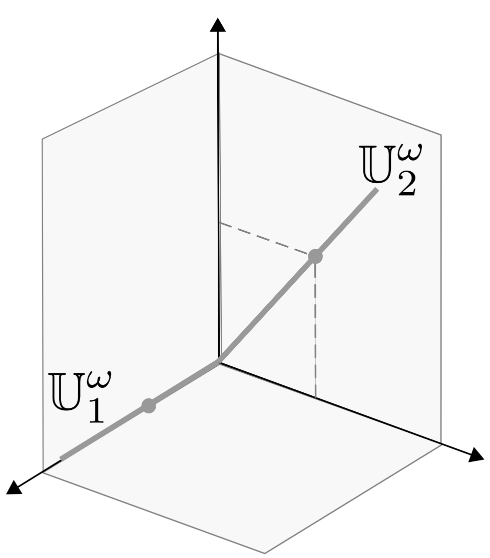

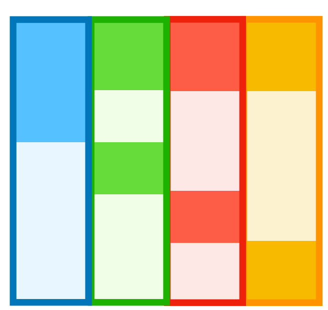

Question: Which projections do we need to observe for a unique reconstruction?

Previous Work - Noiseless Case

Theorem 1 - (Pimentel-Alarcón, Nowak, Boston, ISIT '15)

$\mathbb{U}$ can be recovered from $\{\mathbb{U}_i^\omega\}'$s if and only if $\{\mathbb{U}_i^\omega\}'$s are observed in the right places*. Meaning:

Each subset of n projections has at least n+r known entries

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*There are some more technical conditions

- There are N = d - r observations (WLOG)
- Each observation has r+1 entries (WLOG)

But this is the essence of the deterministic result

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More on the construction soon!

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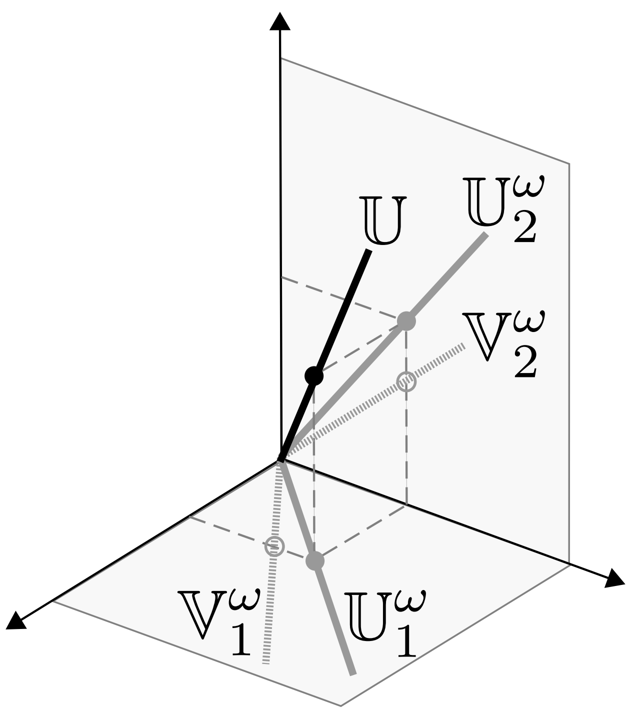

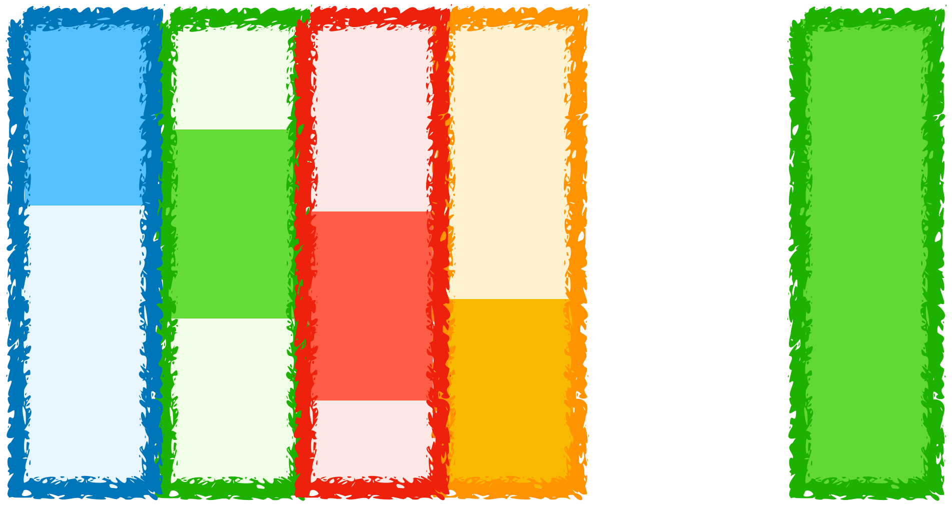

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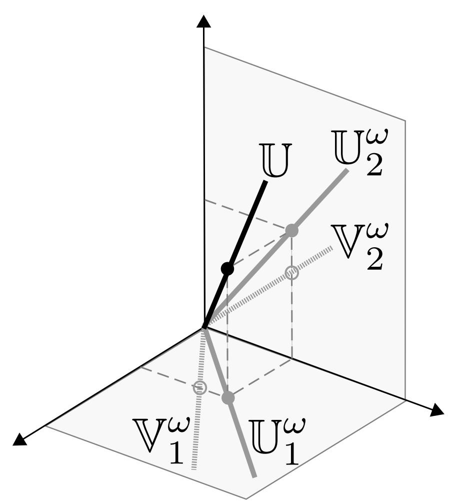

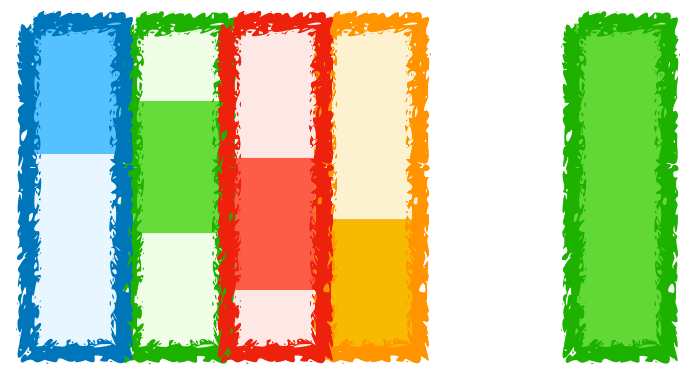

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Technical assumptions:

➛ The sampling satisfies the conditions in Theorem 1 (P.-A., et.al)

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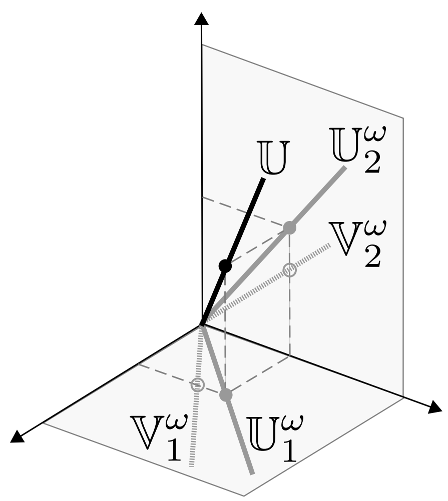

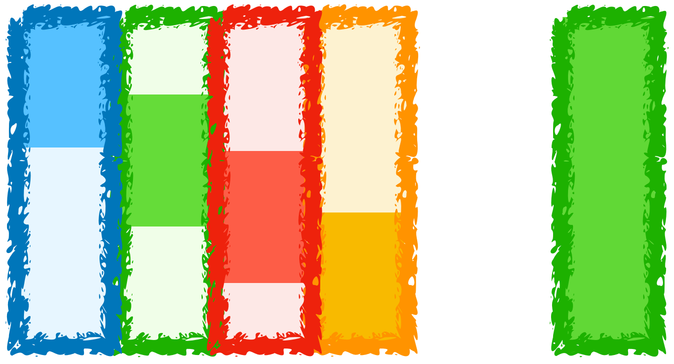

Goal: To estimate $\mathbb{U}$ from the $\{\mathbb{V}_i^\omega\}'$s and bound the error

Technical assumptions:

- The sampling satisfies the conditions in Theorem 1 (P.-A., et.al)
- $|Z_i^\omega|_2 < \epsilon$, so ϵ is the noise
- $\min_i(\sigma(\mathbb{V}_i^\omega)) =: \delta$, and δ is the signal

Main Result

Noisy Data and Estimation Bound

Theorem (S., P.-A., this paper)

For almost every $\mathbb{U}$,

$$d_G(\mathbb{U}, \hat{\mathbb{U}}) \leq \frac{\epsilon\sqrt{2r(d-r)}}{\delta \cdot \sigma(B)}$$

Noisy Data and Estimation Bound

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For almost every $\mathbb{U}$,

$$d_G(\mathbb{U}, \hat{\mathbb{U}}) \leq \frac{\epsilon\sqrt{2r(d-r)}}{\delta \cdot \sigma(B)}$$

This refers to a set of measure zero subspaces that we will not identify.

Noisy Data and Estimation Bound

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For almost every $\mathbb{U}$,

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This is the distance between $\mathbb{U}$ and $\hat{\mathbb{U}}$ on the grassmannian manifold - a standard metric to measure how different two subspaces are.

$$d_G(A, B) = \frac{1}{\sqrt{2}}\|P_A - P_B\|_F$$

Noisy Data and Estimation Bound

Theorem (S., P.-A., this paper)

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$$d_G(\mathbb{U}, \hat{\mathbb{U}}) \leq \frac{\epsilon\sqrt{2r(d-r)}}{\delta \cdot \sigma(B)}$$

This is the (multiplicative inverse of the) *signal-to-noise ratio* where

- $|Z_i^\omega|_2 < \epsilon$, for every i, bounds the *noise*
- $\min_i(\sigma(\mathbb{V}_i^\omega)) =: \delta$ requires that the *signal* power is at least δ

Noisy Data and Estimation Bound

Theorem (S., P.-A., this paper)

For almost every $\mathbb{U}$,

$$d_G(\mathbb{U}, \hat{\mathbb{U}}) \leq \frac{\epsilon\sqrt{2r(d-r)}}{\delta \cdot \sigma(B)}$$

$$\epsilon = 0 \implies d_G(\mathbb{U}, \hat{\mathbb{U}}) = 0 \implies \mathbb{U} = \hat{\mathbb{U}}$$

Taking $\epsilon = 0$, this bound recovers Theorem 1 (P.-A., et. al).

Noisy Data and Estimation Bound

Theorem (S., P.-A., this paper)

For almost every $\mathbb{U}$,

$$d_G(\mathbb{U}, \hat{\mathbb{U}}) \leq \frac{\epsilon\sqrt{2r(d-r)}}{\delta \cdot \sigma(B)}$$

This number is the number of degrees of freedom of an $r-$dimensional subspace of $\mathbb{R}^d$.

Noisy Data and Estimation Bound

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For almost every $\mathbb{U}$,

$$d_G(\mathbb{U}, \hat{\mathbb{U}}) \leq \frac{\epsilon\sqrt{2r(d-r)}}{\delta \cdot \sigma(B)}$$

B is a matrix that encodes the sampling and the orientation of the subspace (more later). $\sigma(B)$ is its smallest singular value.

Noisy Data and Estimation Bound

Computability: The bound is not just theoretical but very computable (i.e. it is based on the observed noisy data)

Theorem (S., P.-A., ISIT 2022)

For almost every $\mathbb{U}$,

$$d_G(\mathbb{U}, \hat{\mathbb{U}}) \leq \frac{\epsilon\sqrt{2r(d-r)}}{\delta \cdot \sigma(B)}$$

Signal-to-noise ratio

Constructed from kernel of noisy projections (Observed data)

Degrees of freedom of r - dimensional subspaces

Sketch of Construction and Proof

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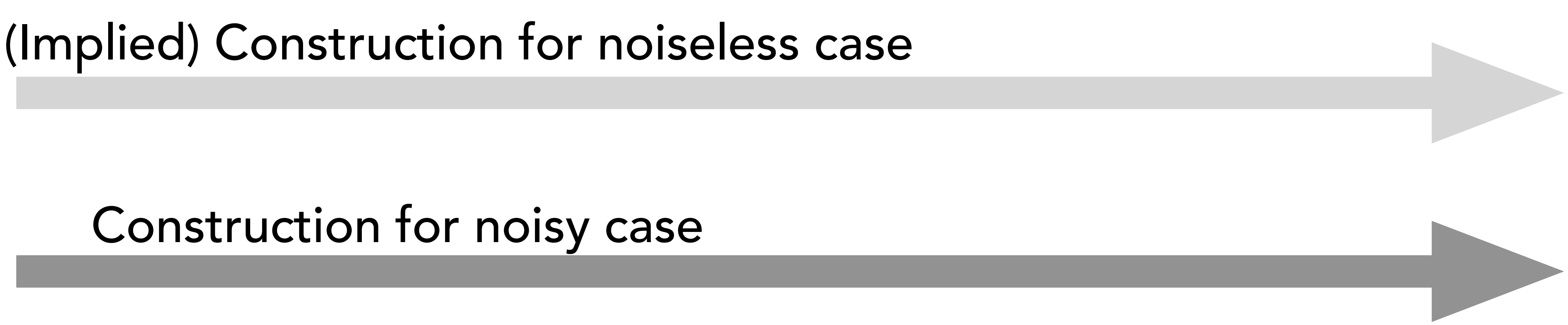
Gameplan

Construction for noisy case

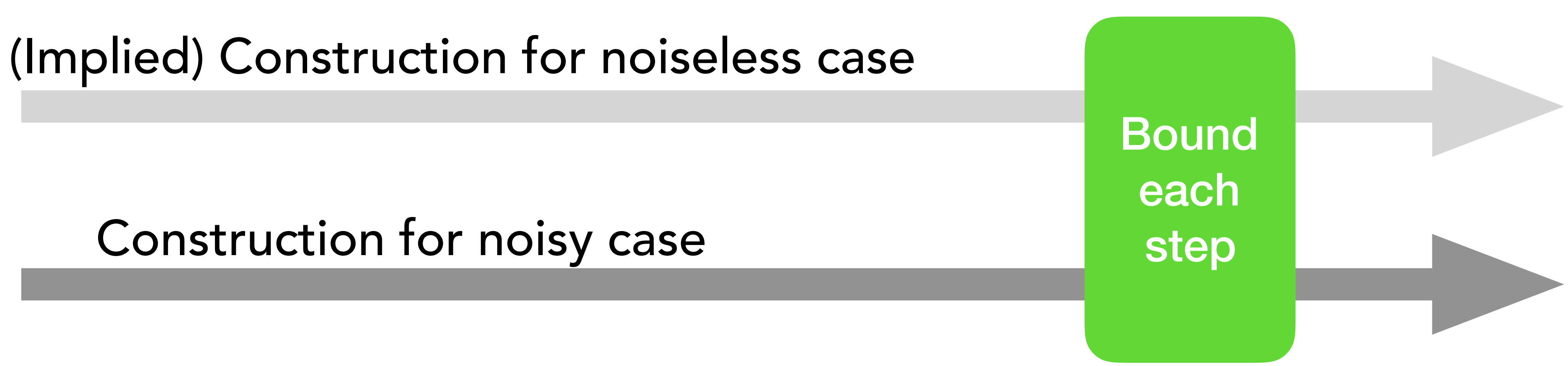
Sketch of Construction and Proof

Gameplan



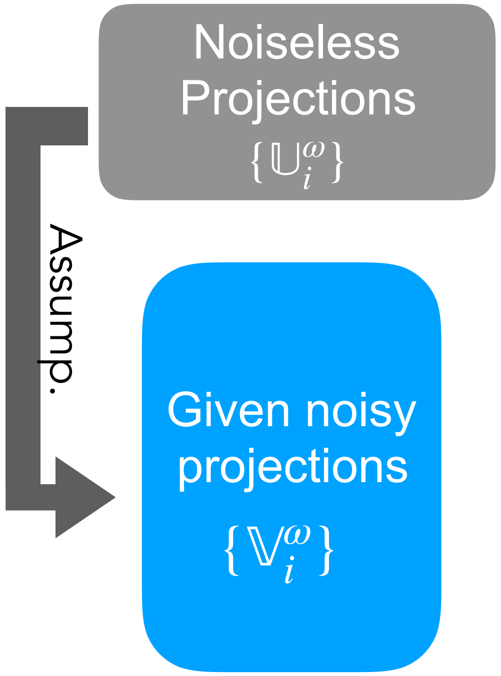
Sketch of Construction and Proof

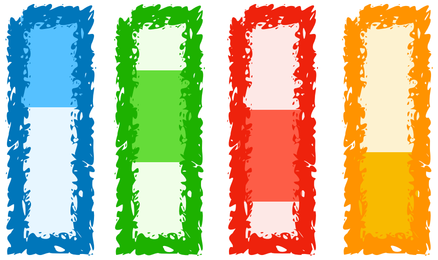
Gameplan



ROLL UP YOUR SLEEVES!

Noisy Data and Estimation Bound





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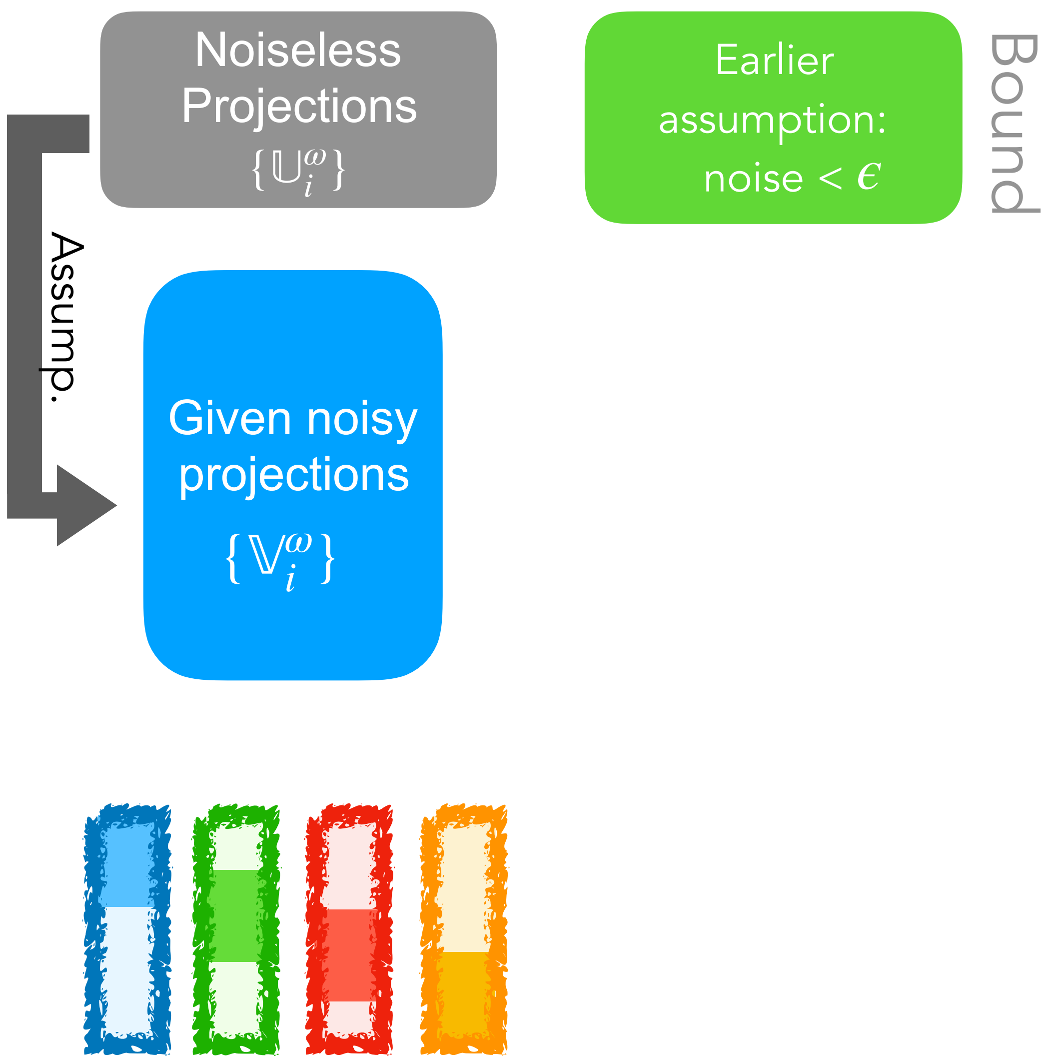

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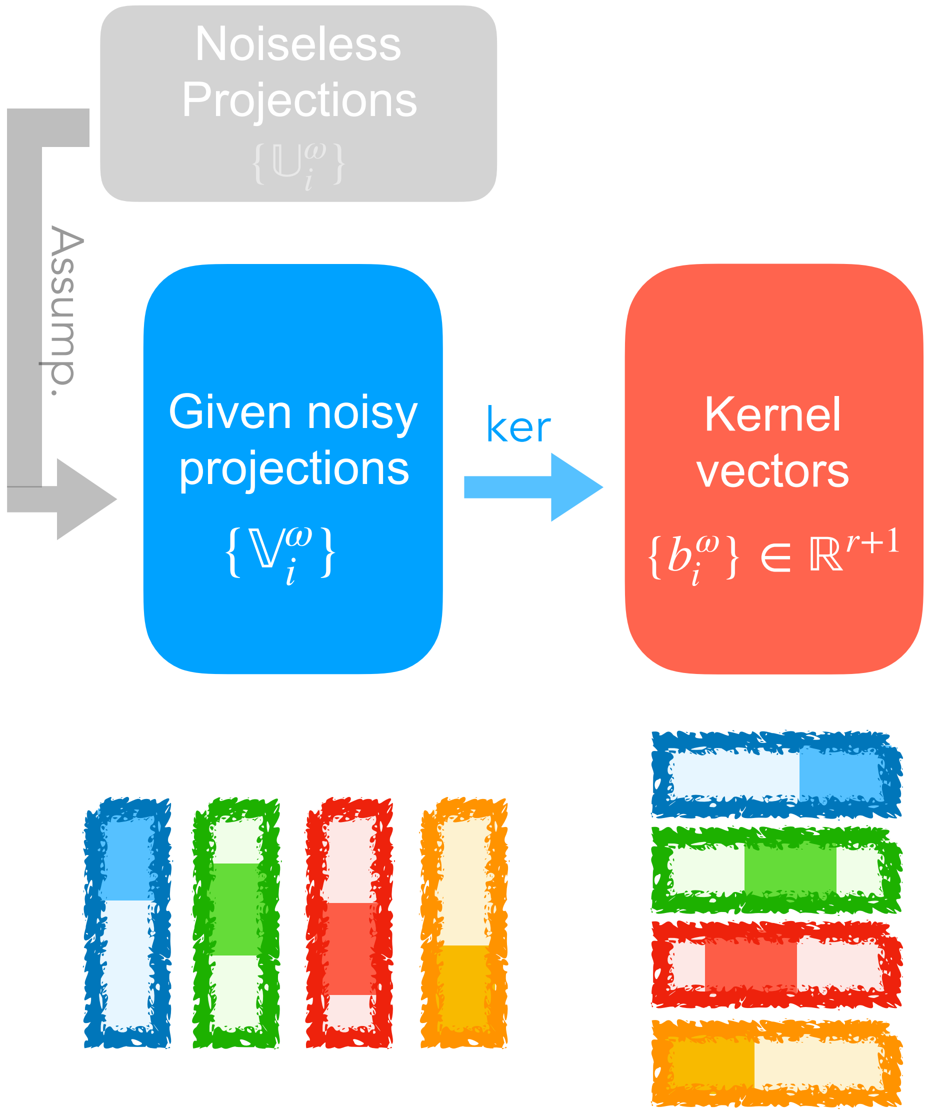
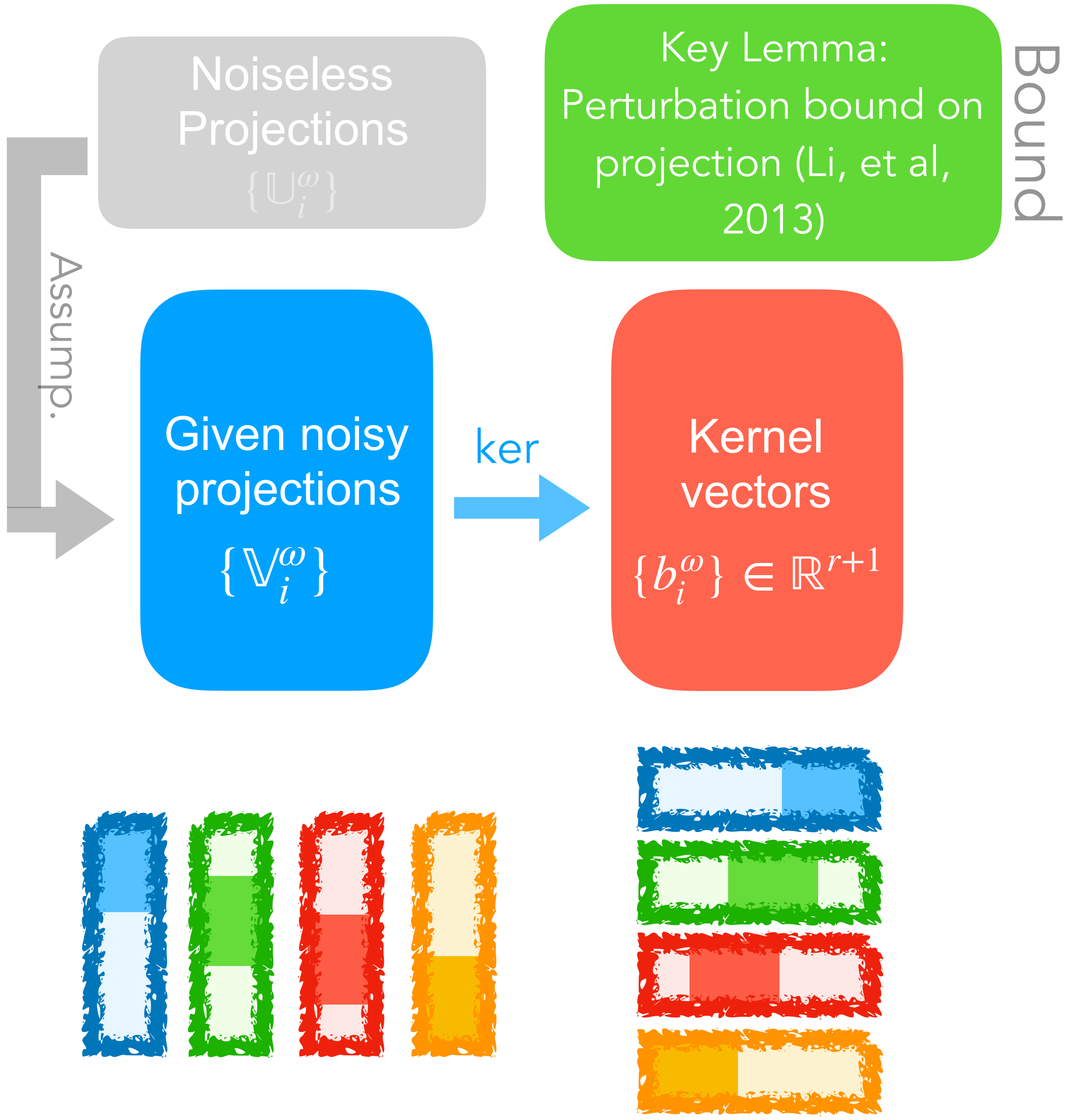


Step 1: Take a normal vector $b_i^\omega \in \ker \mathbb{V}_i^\omega \subset \mathbb{R}^{d+1}$ for each projection

Noisy Data and Estimation Bound



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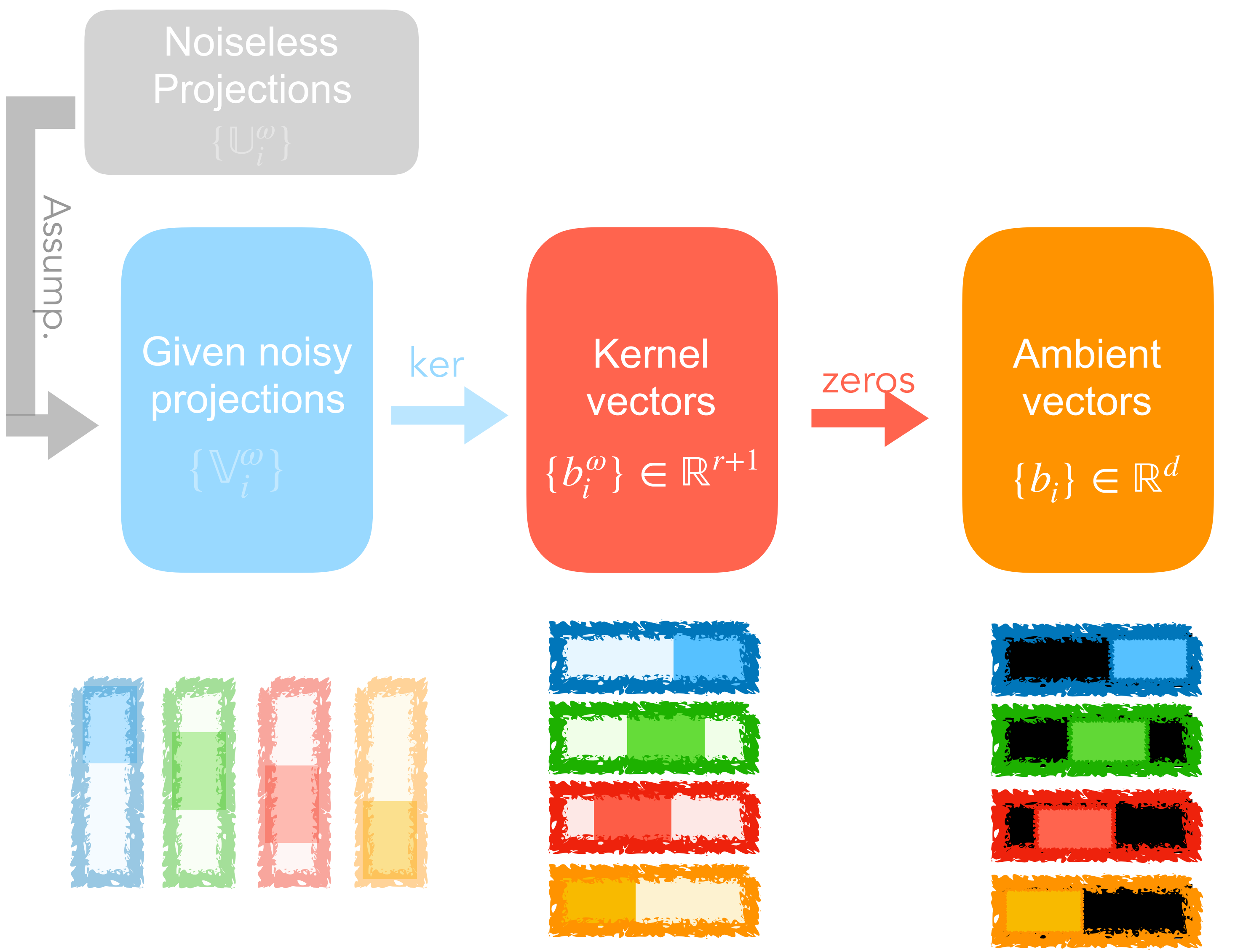

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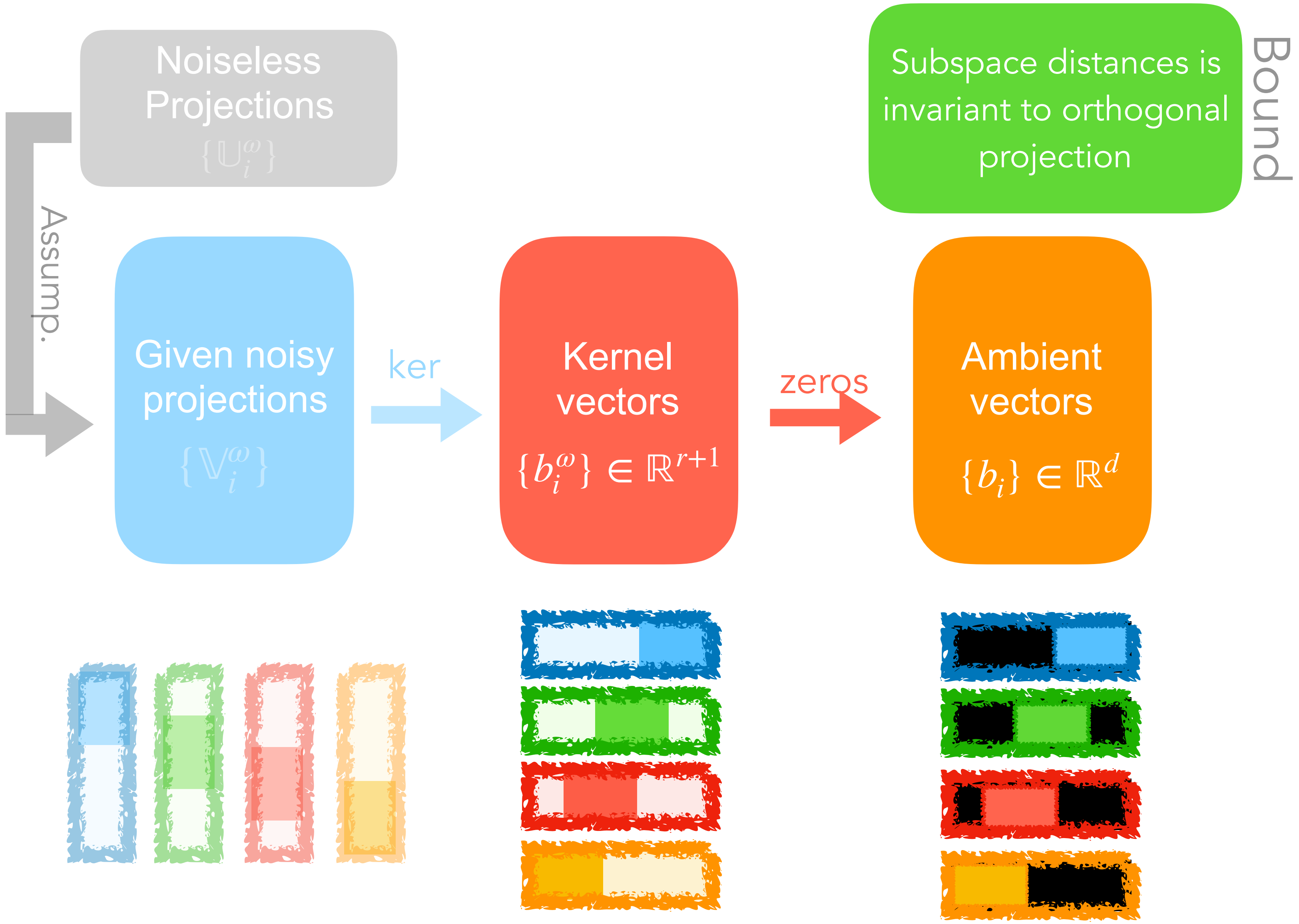
Step 2: Construct vectors $b_i \in \mathbb{R}^d$ by padding b_i^ω with zeros where the sampling is zero

Noisy Data and Estimation Bound



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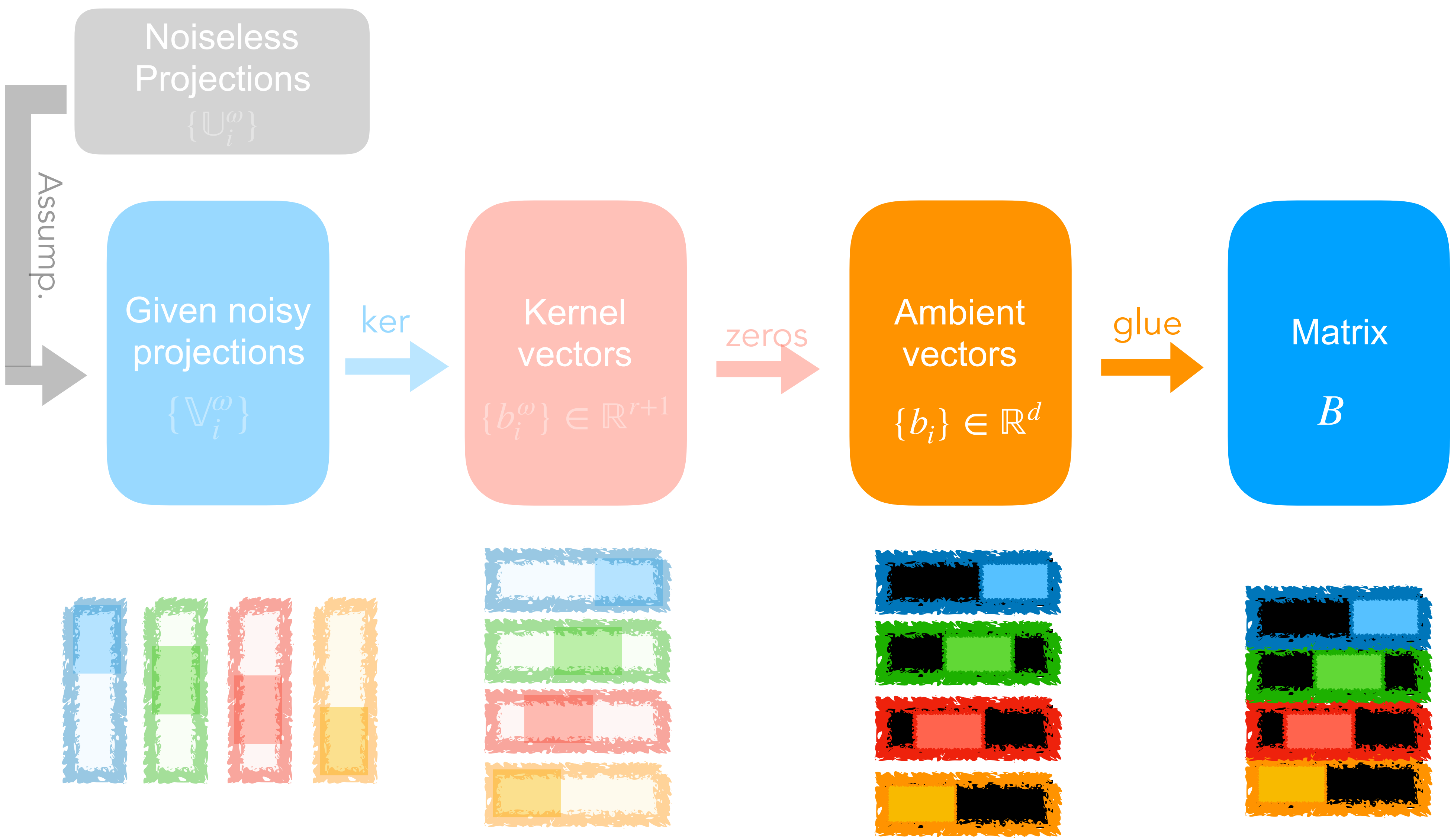

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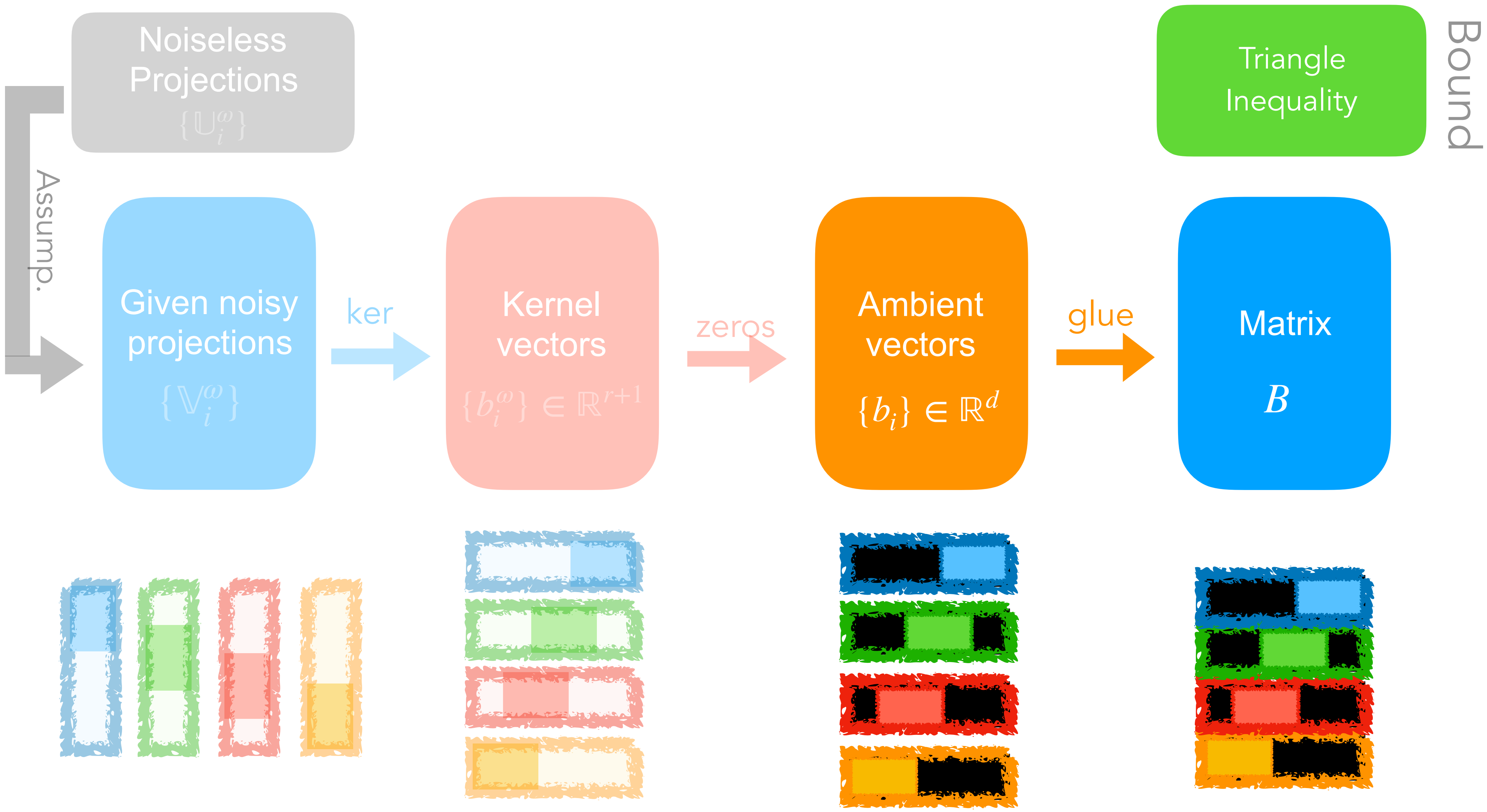

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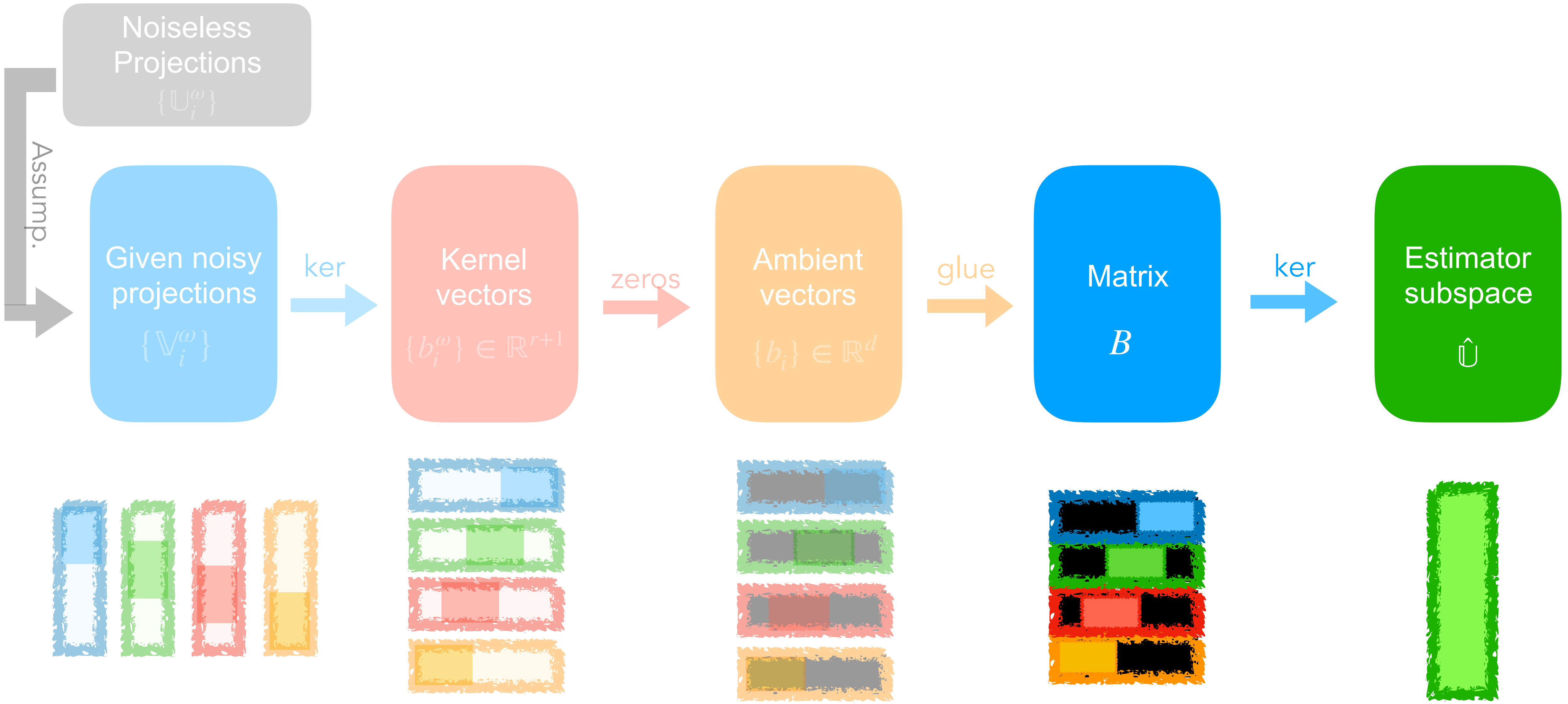

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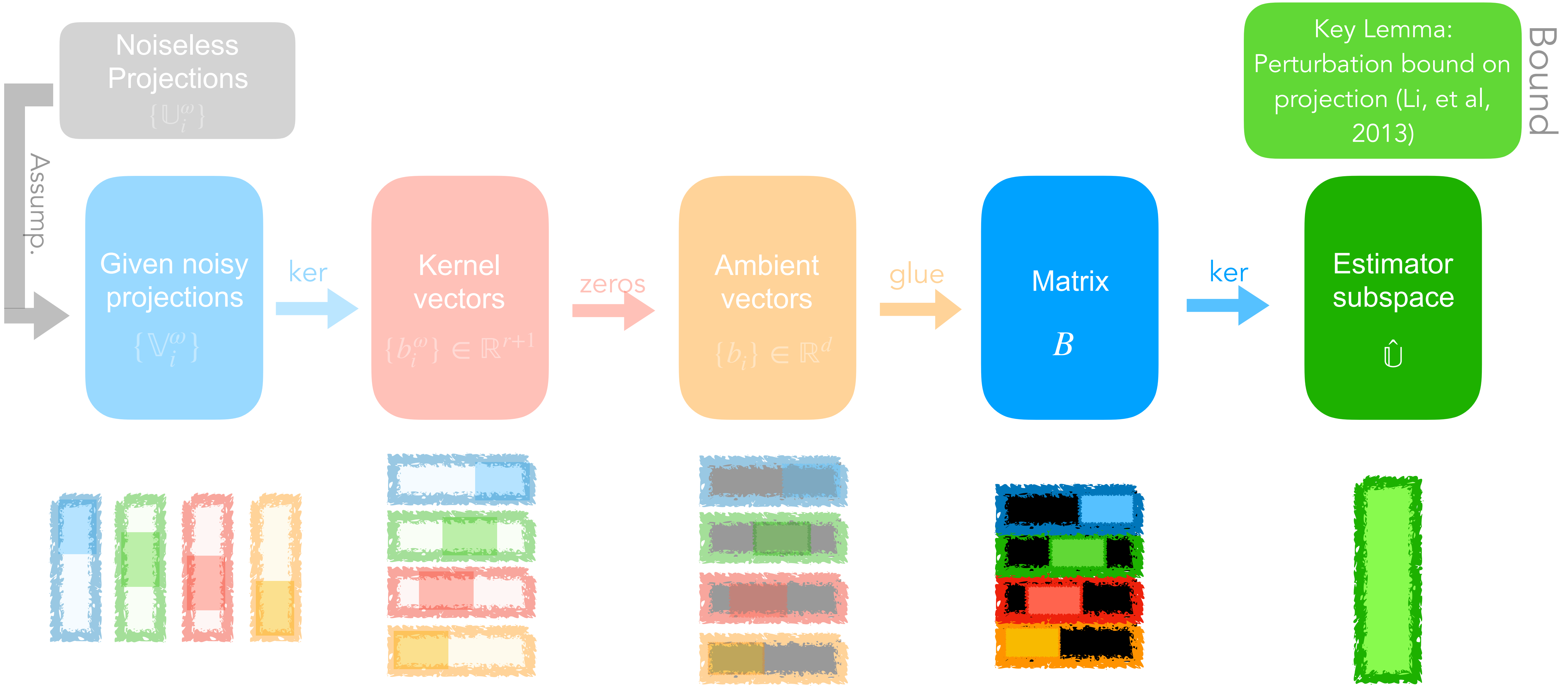

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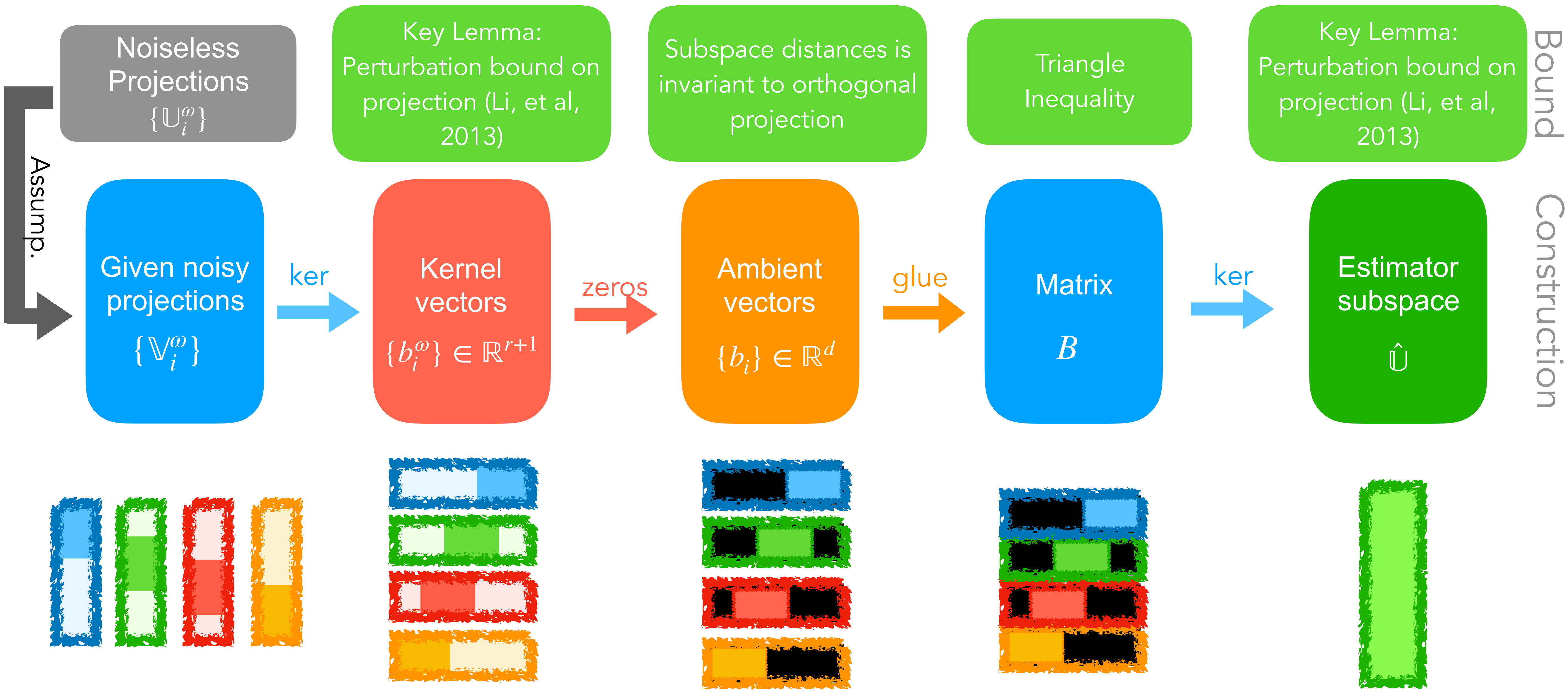

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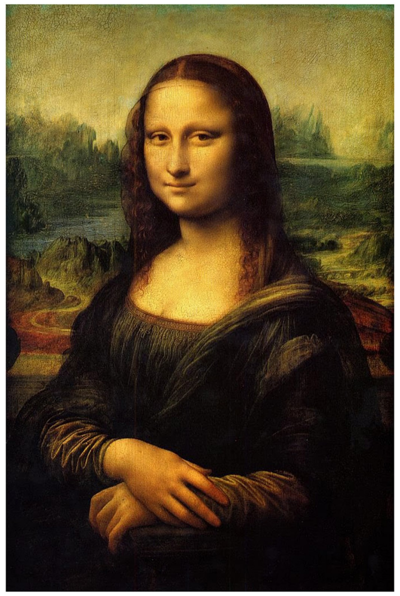
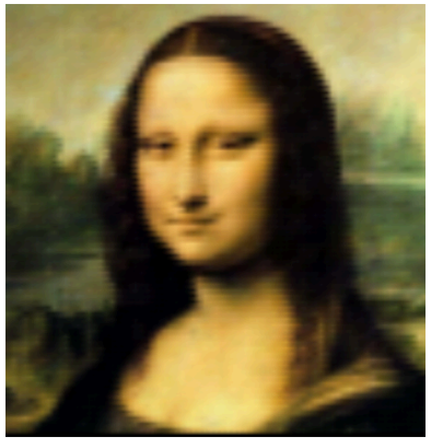
Noisy Data and Estimation Bound

Okay, but **how good is the estimator**?

True Space $\mathbb{U}$



Estimator $\hat{\mathbb{U}}$



or

?

Noisy Data and Estimation Bound

Theorem (S., P.-A., this paper)

For almost every $\mathbb{U}$,

$$d_G(\mathbb{U}, \hat{\mathbb{U}}) \leq \frac{\epsilon\sqrt{2r(d-r)}}{\delta \cdot \sigma(B)}$$

What affects this?

Noisy Data and Estimation Bound

$\sigma(B)$

Noisy Data and Estimation Bound

$\sigma(B)$

Residual of projecting
columns of B
onto each other

Noisy Data and Estimation Bound

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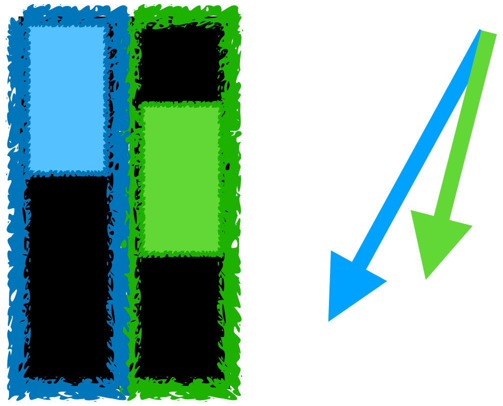

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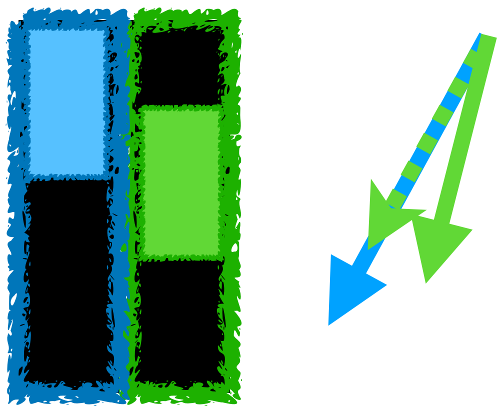

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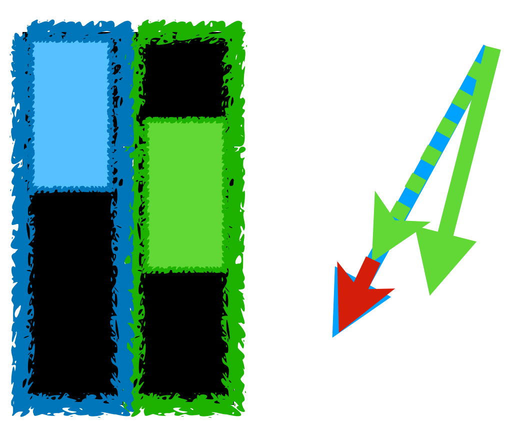

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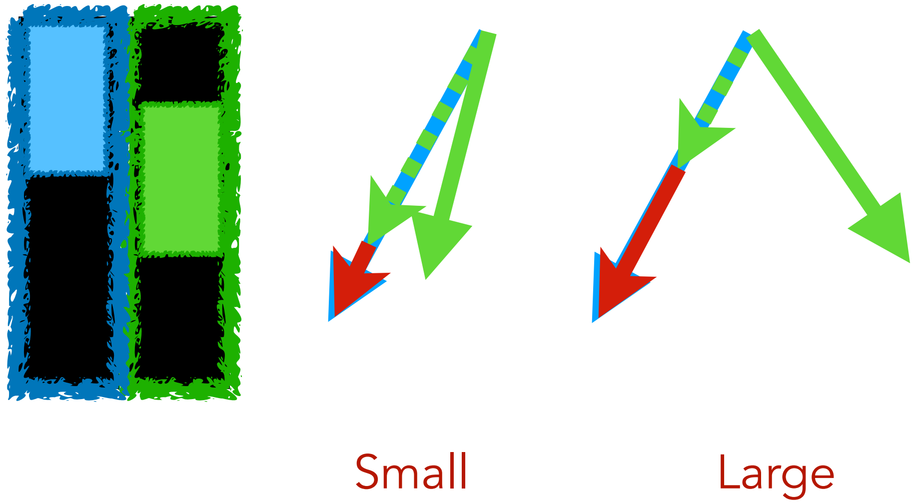

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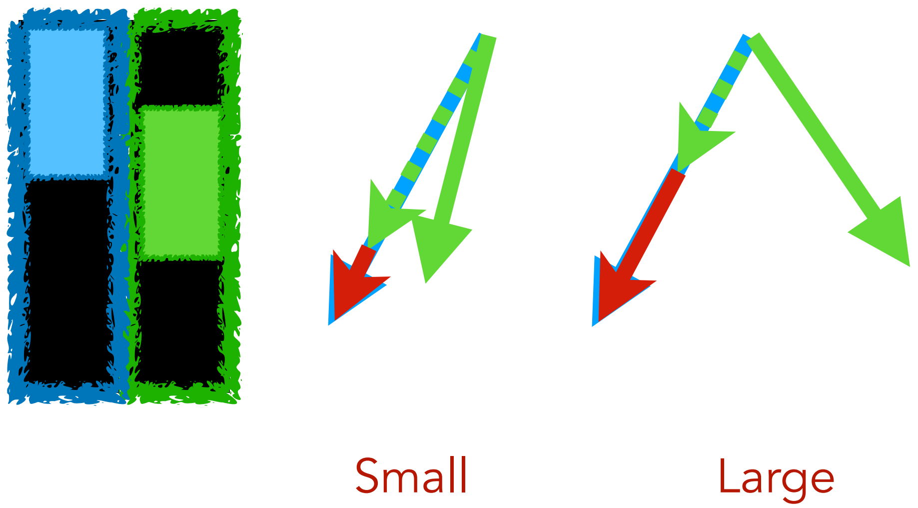
Orientation of Projections $\mathbb{U}_i^{\omega}/\mathbb{V}_i^{\omega}$



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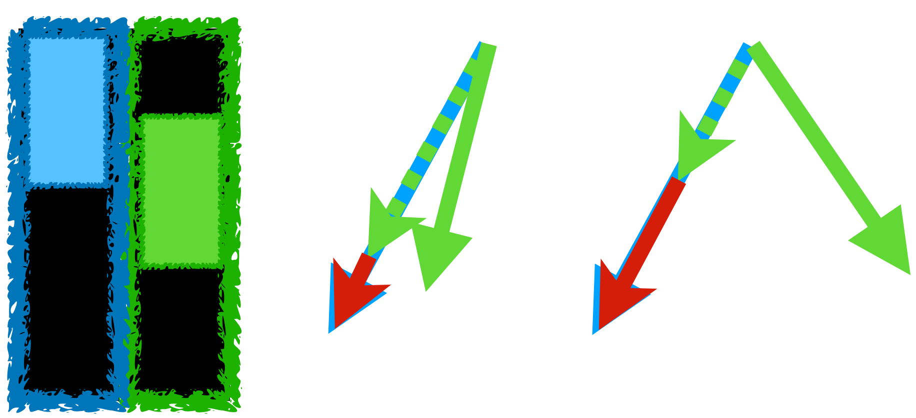

Small

Large

$$\mathbb{U}_i^\omega = \left[\begin{array}{c} \mathbf{I} \\ \hline \alpha_i^T \end{array} \right] \begin{array}{l} \}\, \mathrm{r} \\ \}\, 1 \end{array}$$

Noisy Data and Estimation Bound

$\sigma(B)$

Residual of projecting columns of B onto each other

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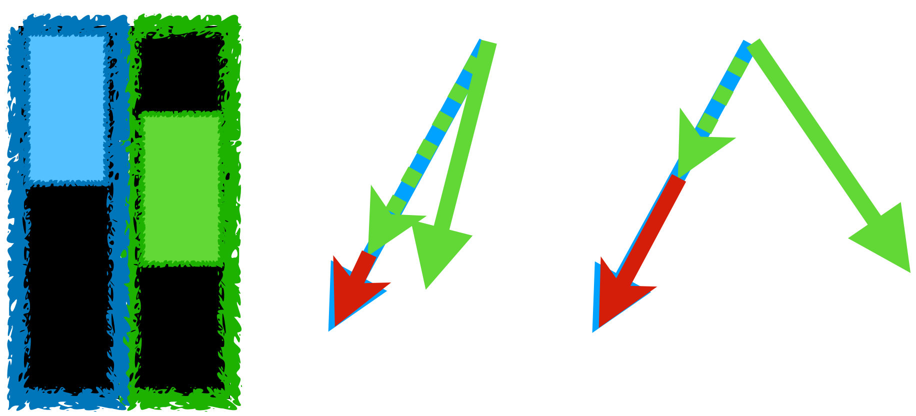

Small

Large

$$\mathbb{U}_i^{\omega} = \begin{bmatrix} \mathbf{I} \\ \hline \alpha_i^T \end{bmatrix} \begin{matrix} \}\, r \\ \}\, 1 \end{matrix}$$

b_i is a perturbation of $[\alpha_i^T, -1]$

Noisy Data and Estimation Bound

Where does this come up?

Noisy Data and Estimation Bound

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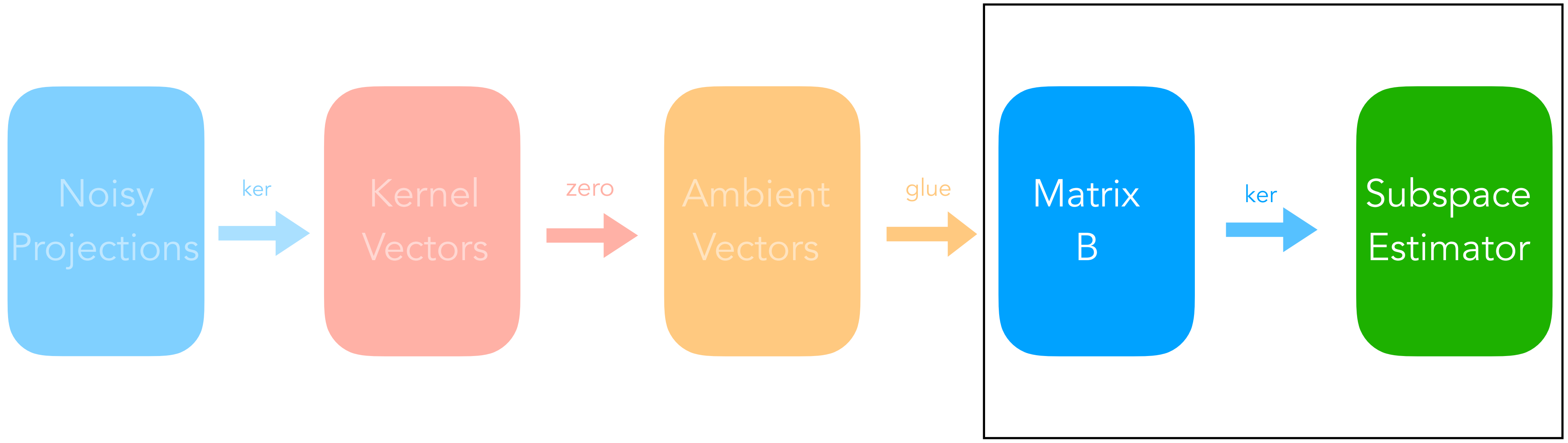

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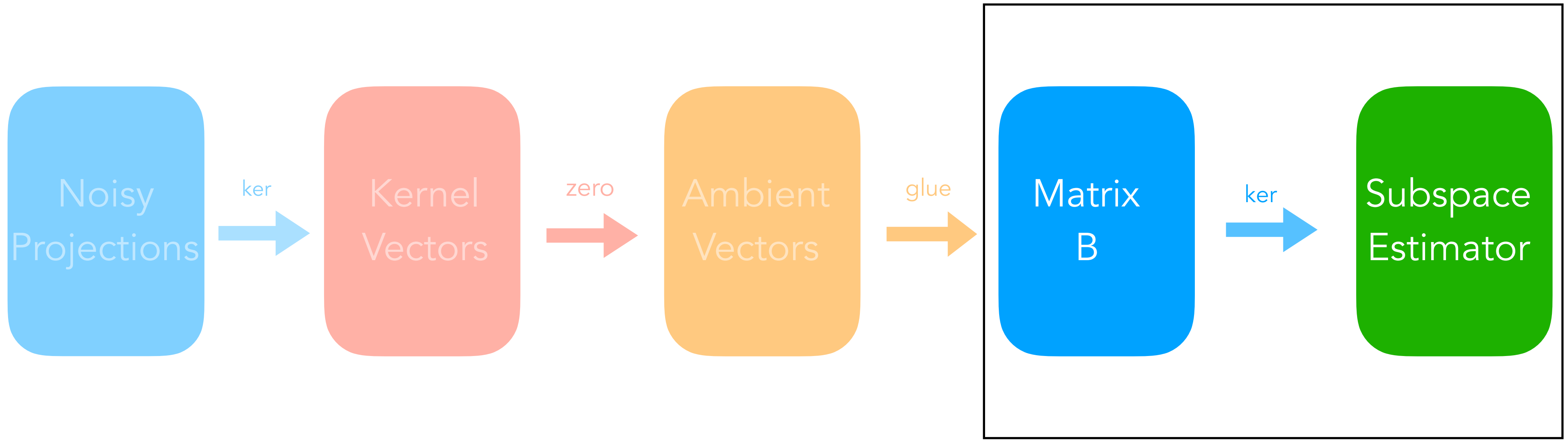

Here, we are taking an orthogonal space

Noisy Data and Estimation Bound

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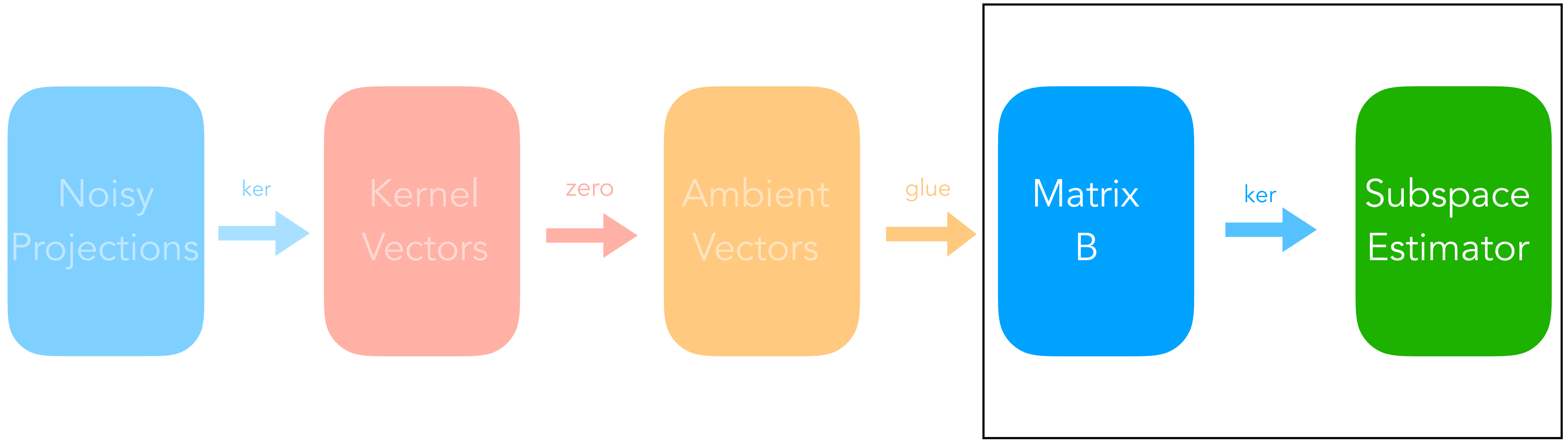

Here, we are taking an orthogonal space
How sensitive is this to perturbations?

Noisy Data and Estimation Bound

How much a subspace gets perturbed depends on how singular its basis is.

Noisy Data and Estimation Bound

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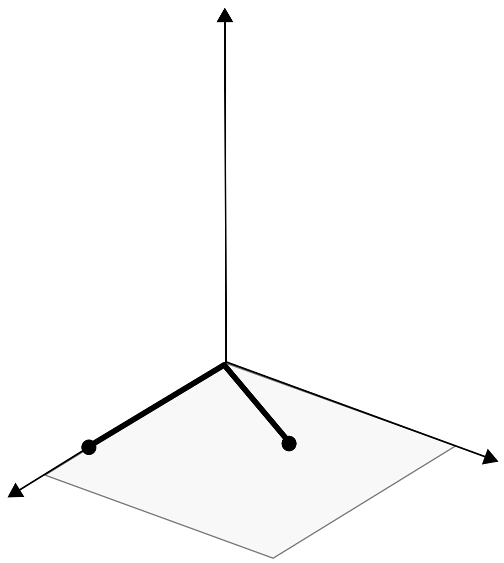
Non-singular Basis



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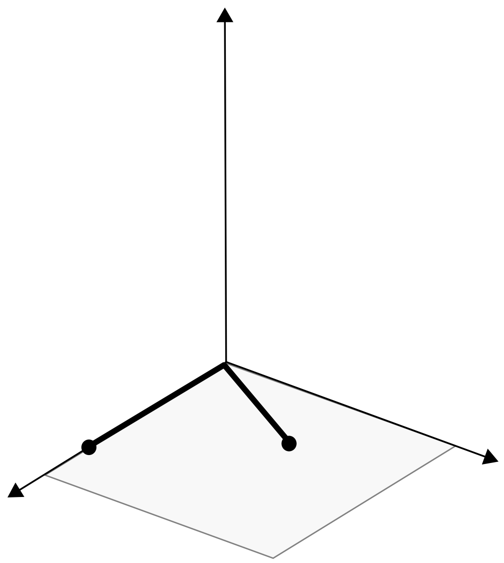

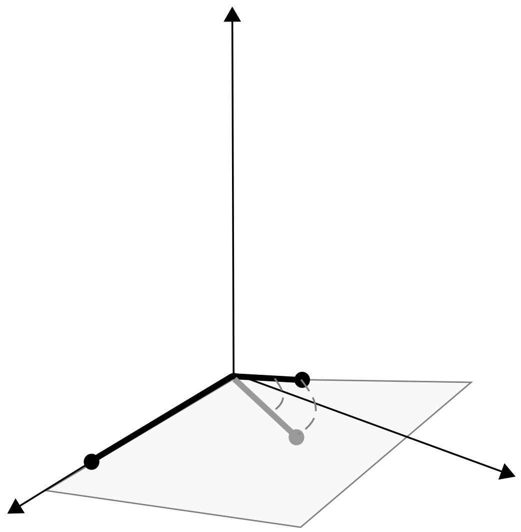

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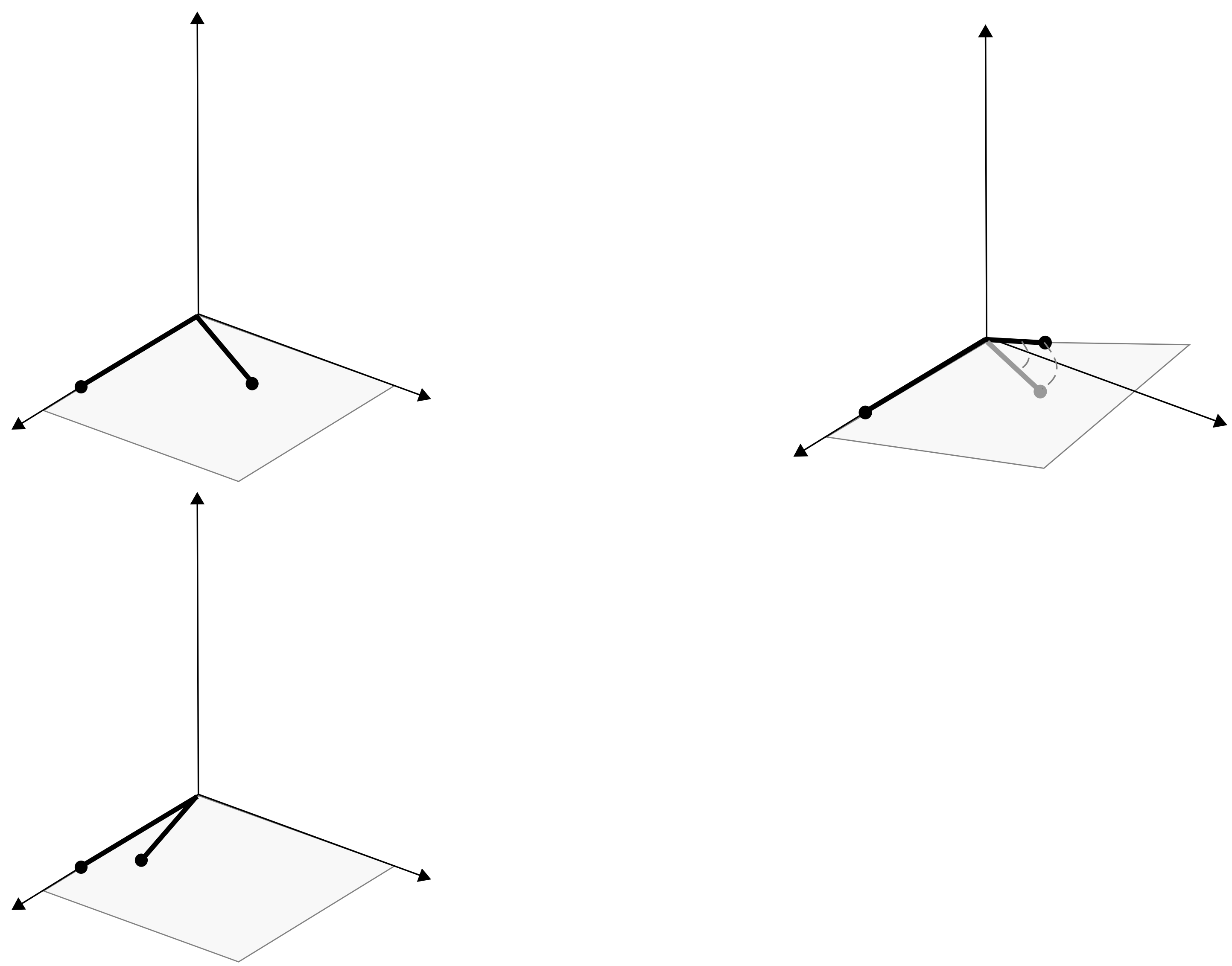
Singular Basis



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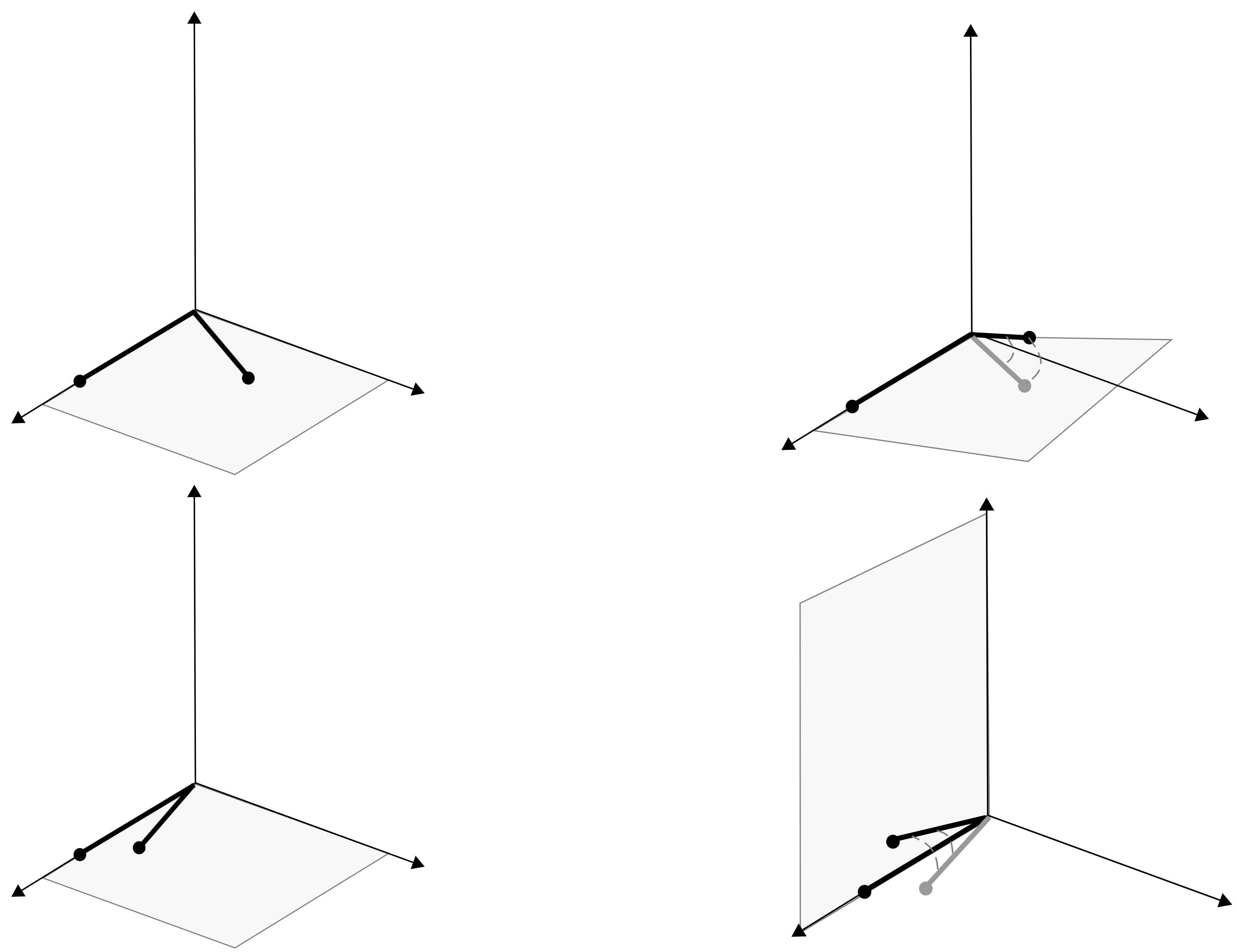

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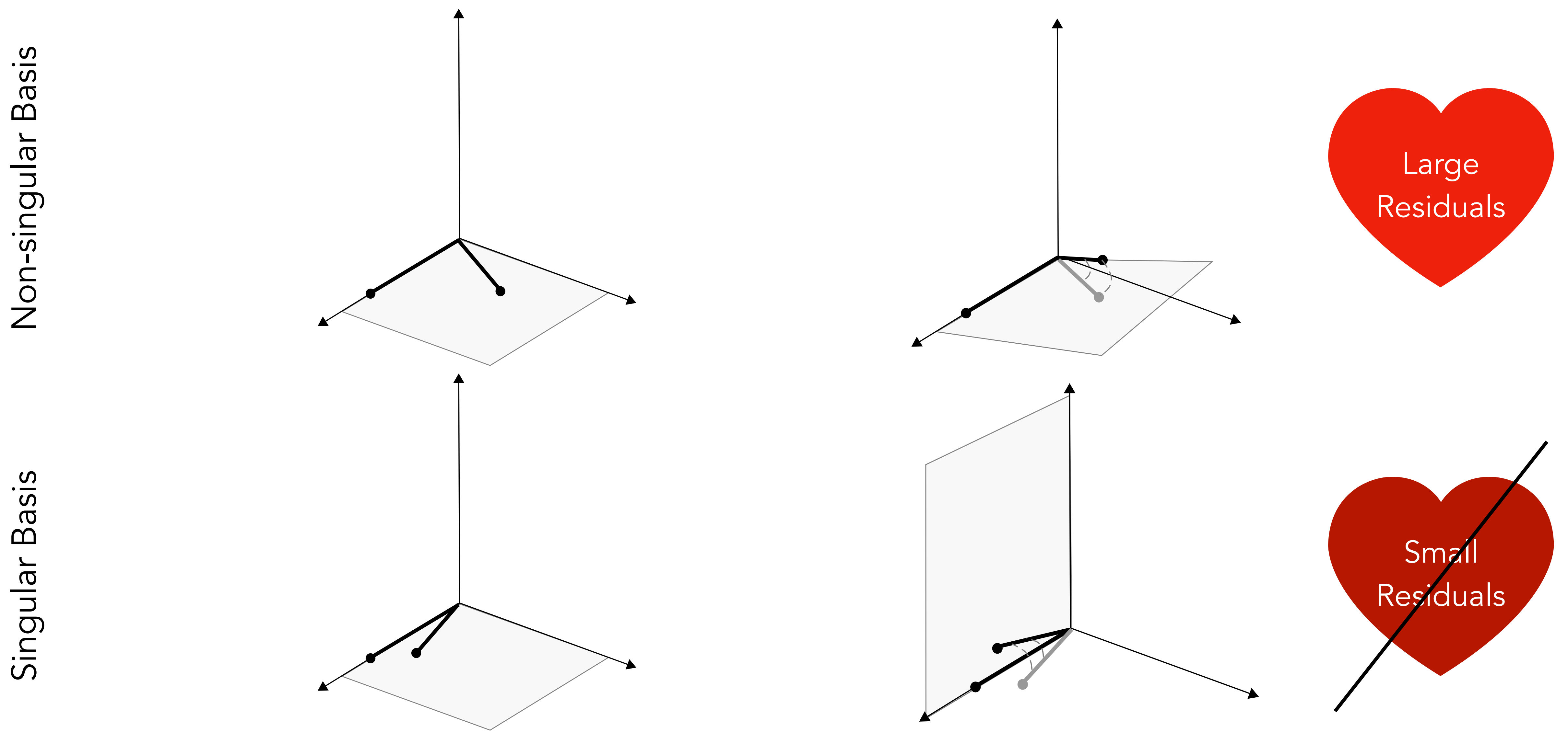

Noisy Data and Estimation Bound

Bottom Line: Different Samplings can lead to different bounds.

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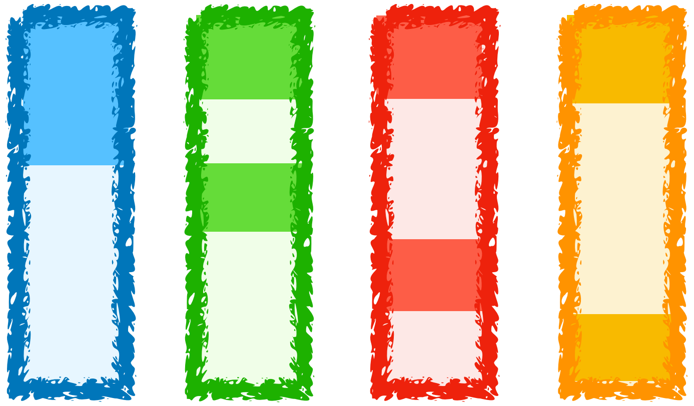
$$\Omega_1 =$$



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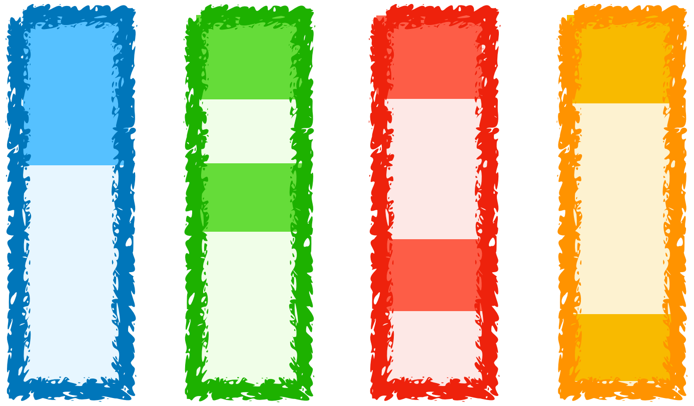

- More overlaps, but
- Each column has a unique coordinate
- Less likely to have small residuals

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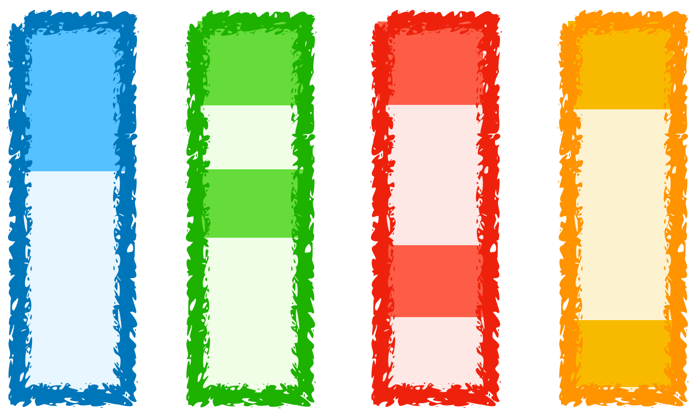

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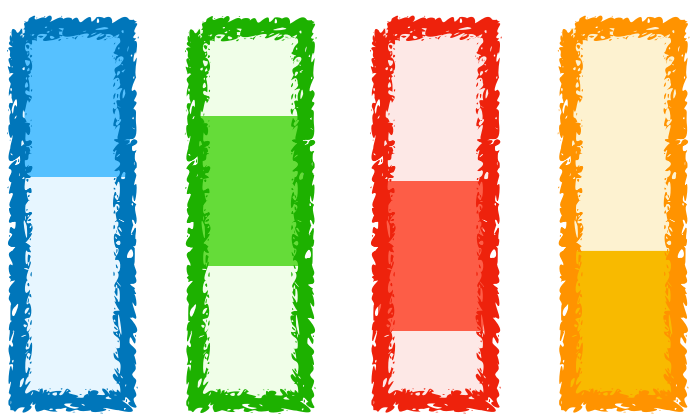

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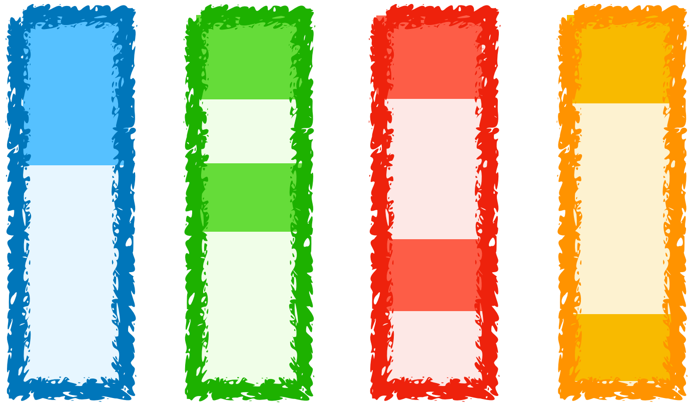

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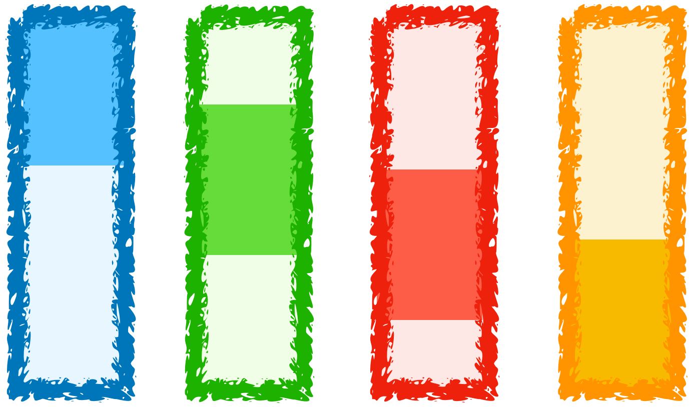

- Fewer overlaps, but
- Most columns don't have unique coordinates
- More likely to have small residuals (than sampling 1)

Noisy Data and Estimation Bound

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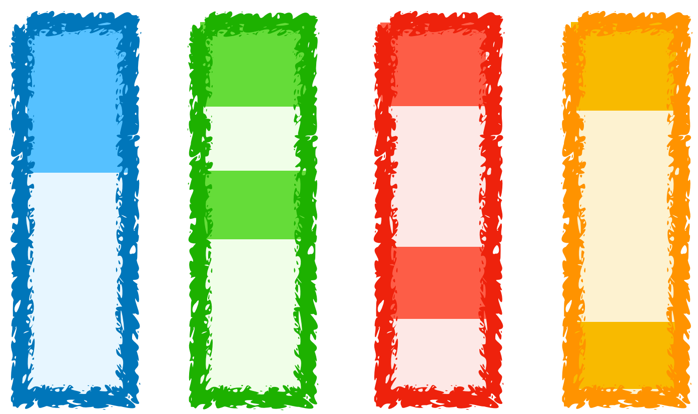

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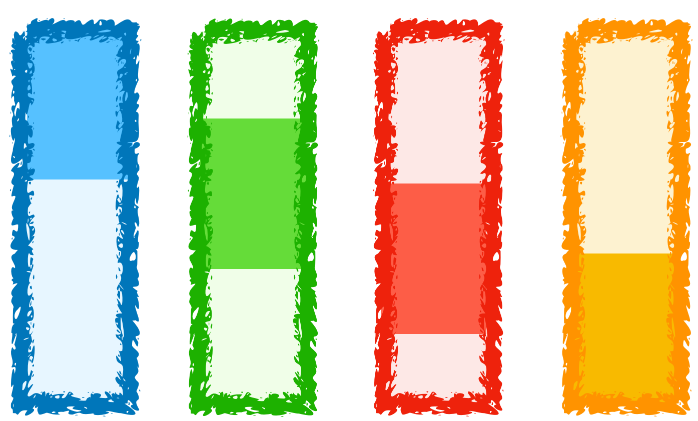

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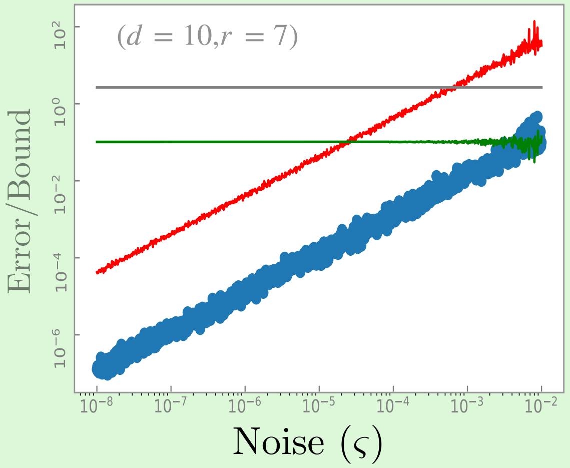
Noisy Data and Estimation Bound

Sampling $\mathbf{\Omega}_1$



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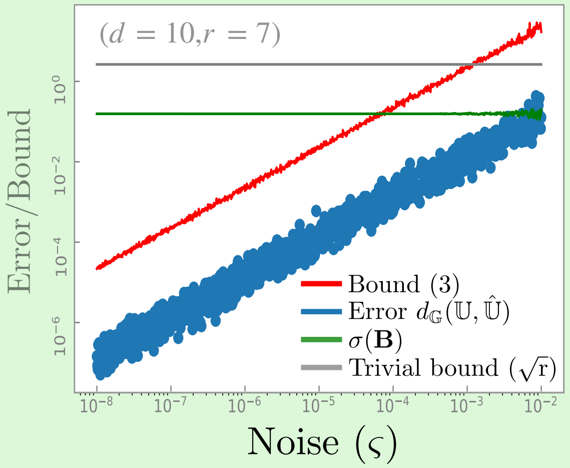

Low dimensions

+

Varying Noise

Bound tight - ✓

Noisy Data and Estimation Bound

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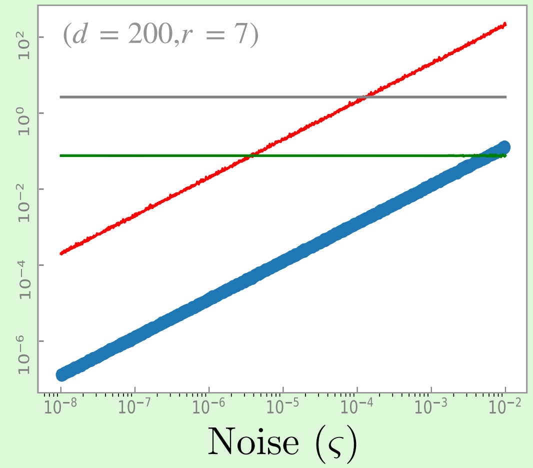

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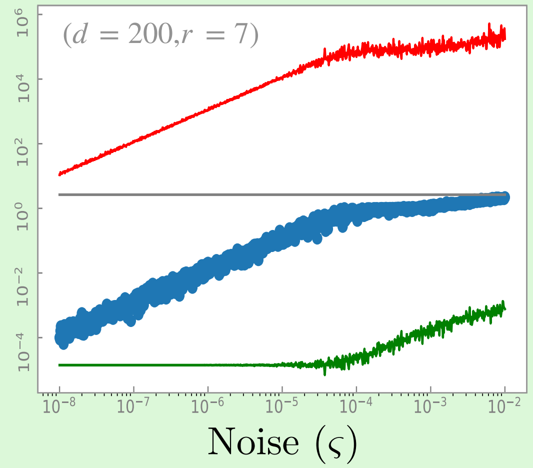


High dimensions
+
Varying Noise

Bound tight - ✓

Depends on Sampling Pattern

Noisy Data and Estimation Bound

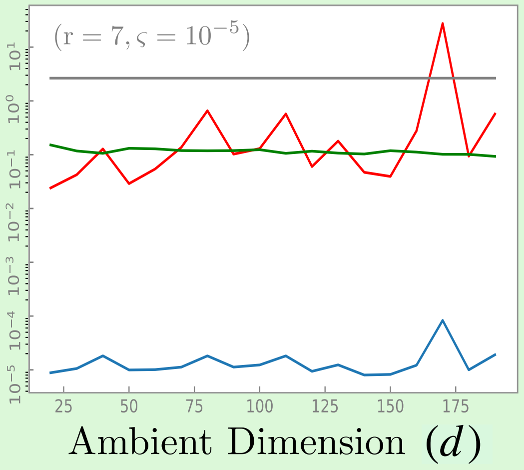
Sampling $\boldsymbol{\Omega}_1$

Error/Bound

Low Subspace Rank
+
Varying Ambient Dim

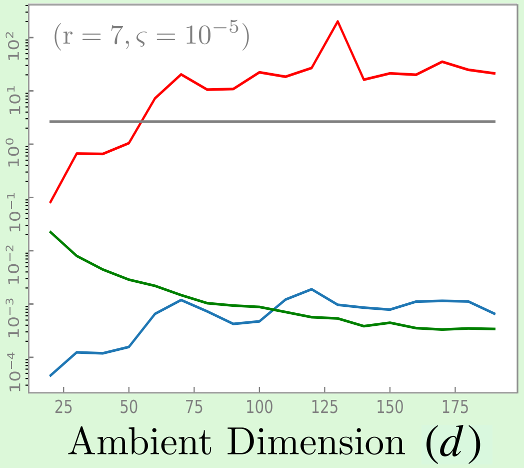
Bound tight - ✓

Depends on sampling.
Good sampling shows bound follow error



Sampling $\boldsymbol{\Omega}_2$

Error/Bound



Noisy Data and Estimation Bound

Sampling $\mathbf{\Omega}_1$

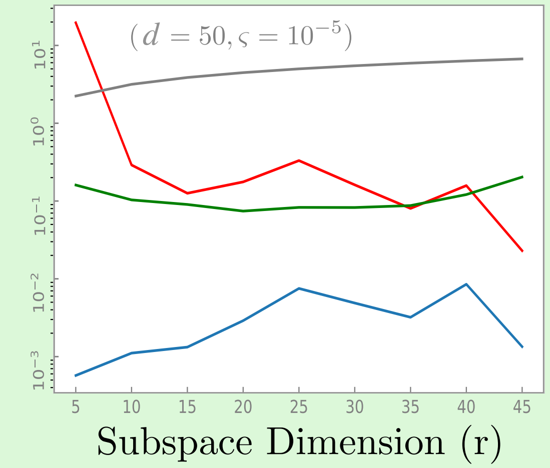
Error/Bound

High ambient dim

+

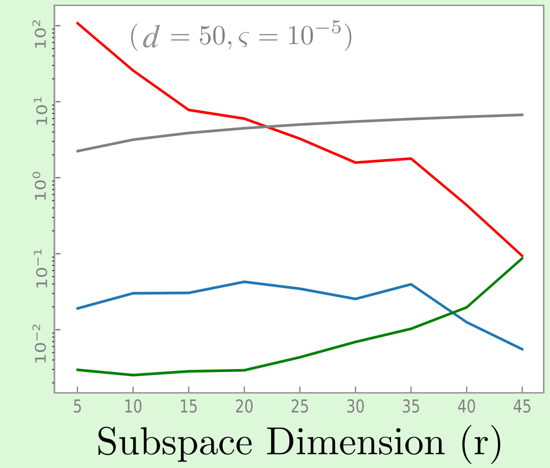
Varying Subspace Dim

Bound tight - ✔



Sampling $\mathbf{\Omega}_2$

Error/Bound



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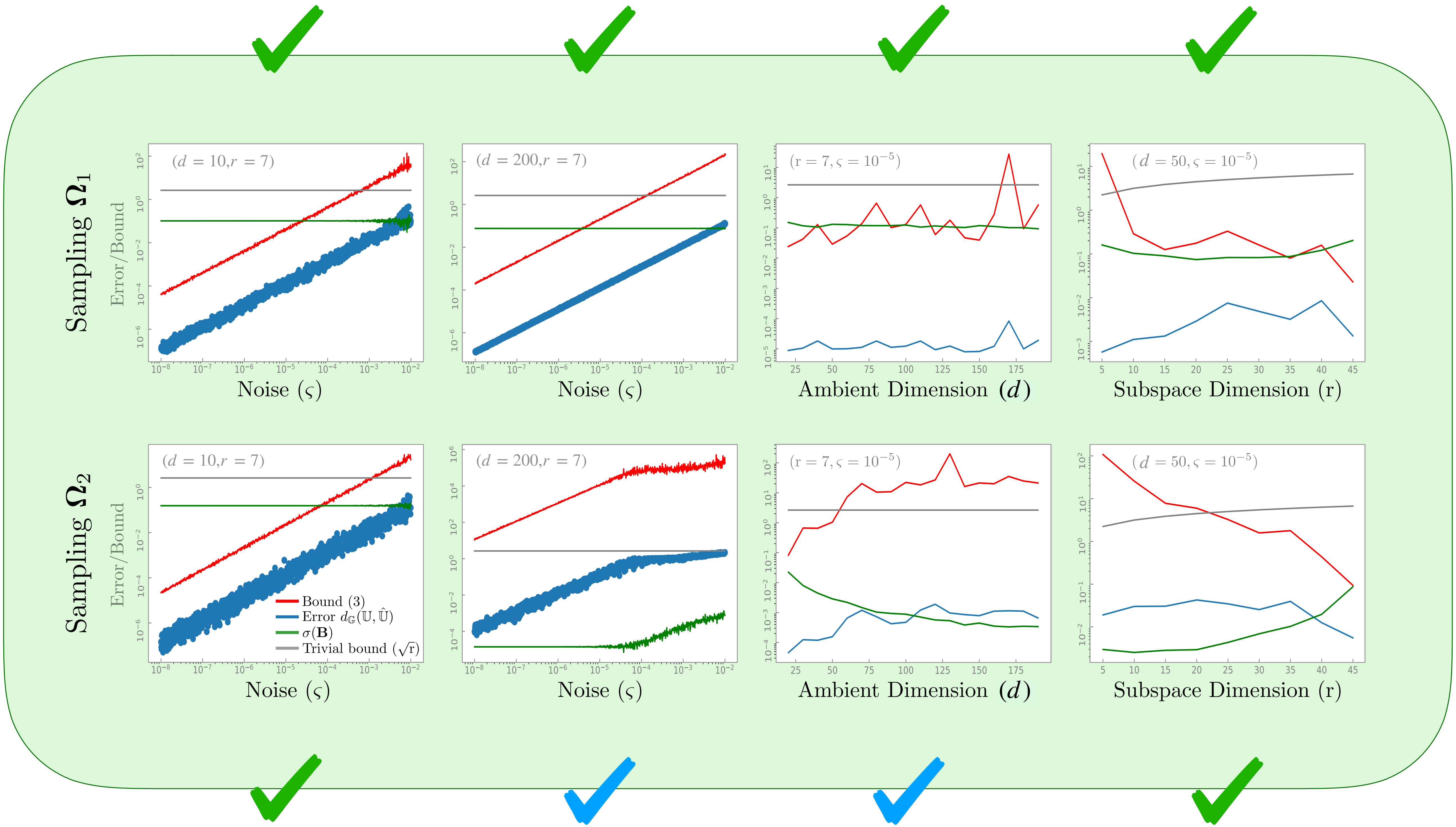

Outline

1. Problem Setup - Subspace Estimation
2. Motivation - Missing Data
3. Previous Work: Noiseless case
4. This Paper - Noisy Data and Estimation Bound
5. Applications
6. Conclusions

Applications - LRMC Theory

Subspace reconstruction has shed light on various LRMC!

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❶ Deterministic sampling conditions for unique completability in LRMC (Theorem 2, Lemma 8 in [2]).
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❸ The fundamental conditions for learning mixtures in MMC (Theorem 1 in [4]).
❹ Identifiability conditions to learn tensorized subspaces in LADMC (Theorem 1 in [5], Lemmas 2, 3 in [6]).
❻ Unique completability conditions for LTRTC (Lemma 4, Theorem 4 in [7], Lemma 9, Theorem 7 in [8]).
❼ Deterministic conditions for unique completability in LCRTC (Lemma 18 in [9]).

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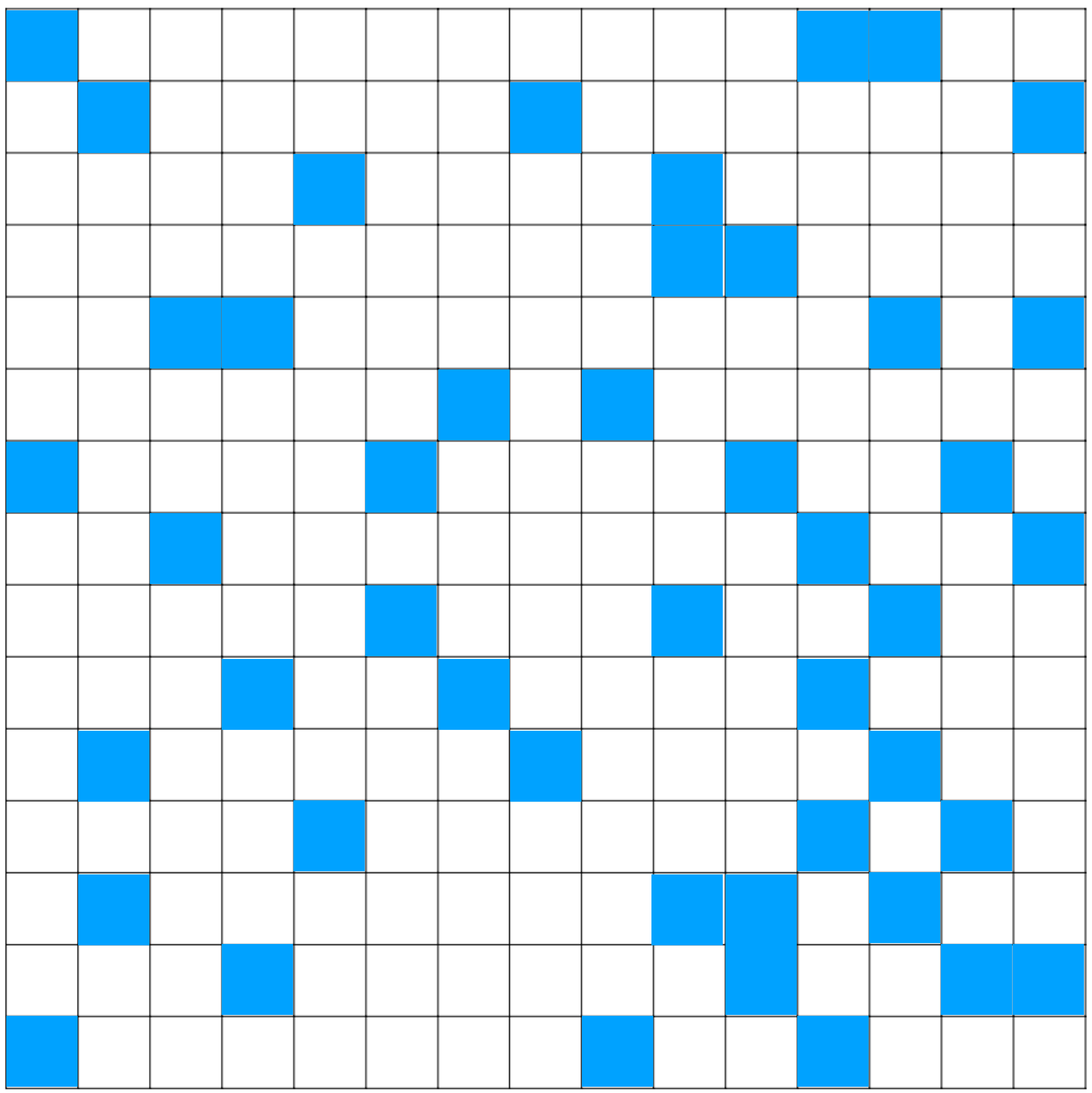
Project incomplete data onto candidate subspaces; reject incompatible spaces.

Applications - RPCA Algorithms



Subspace reconstruction based RPCA Algorithm

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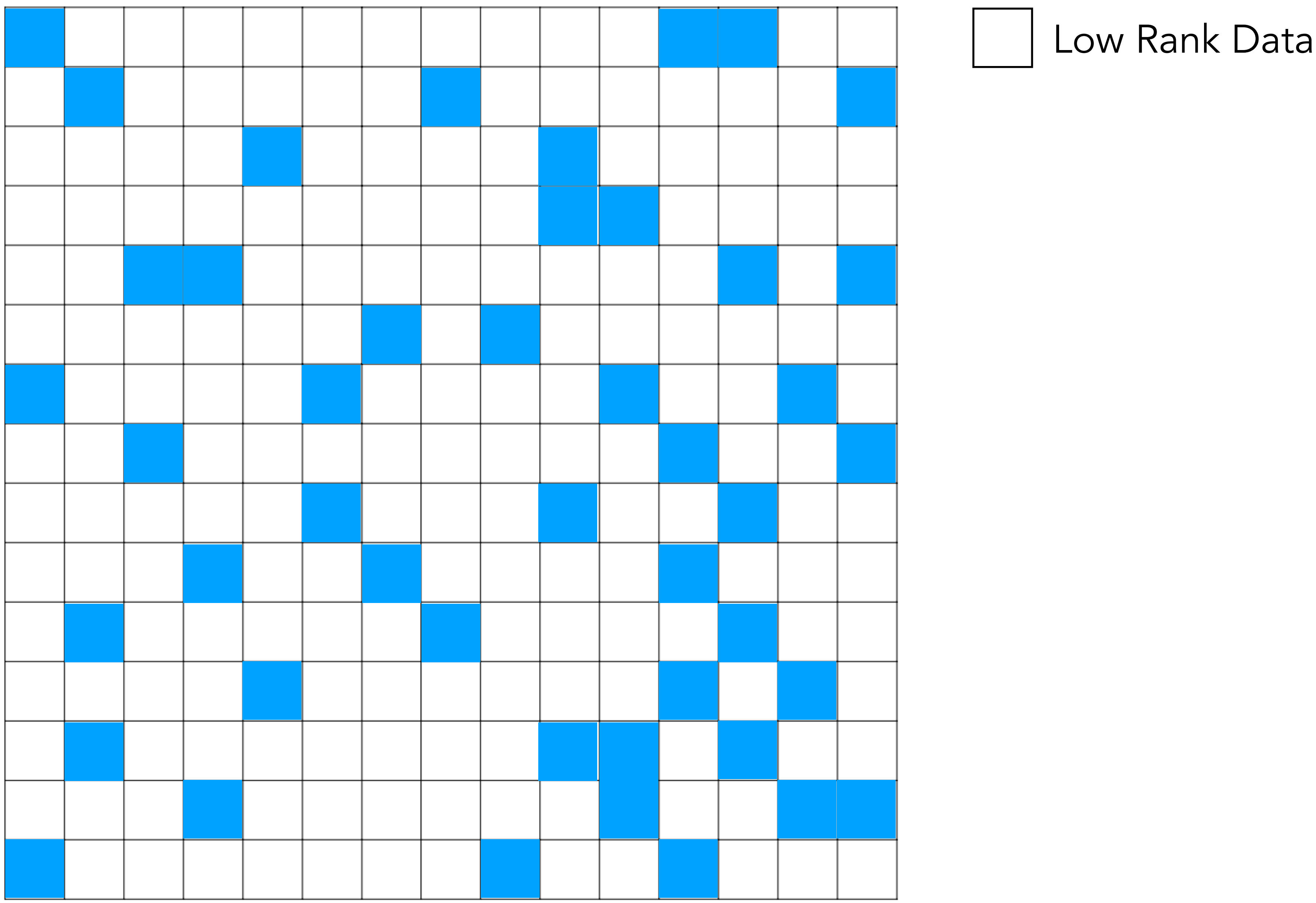

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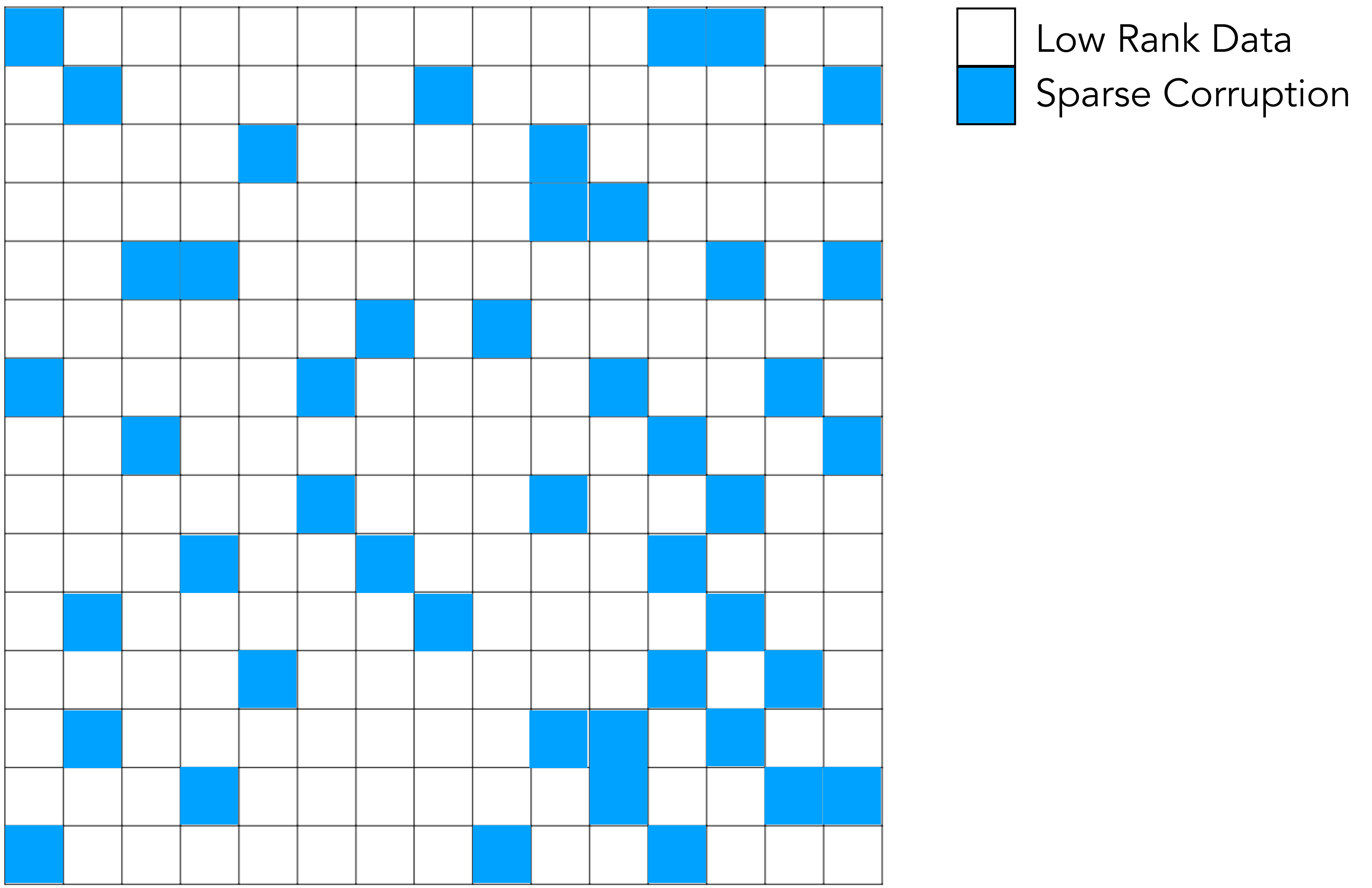

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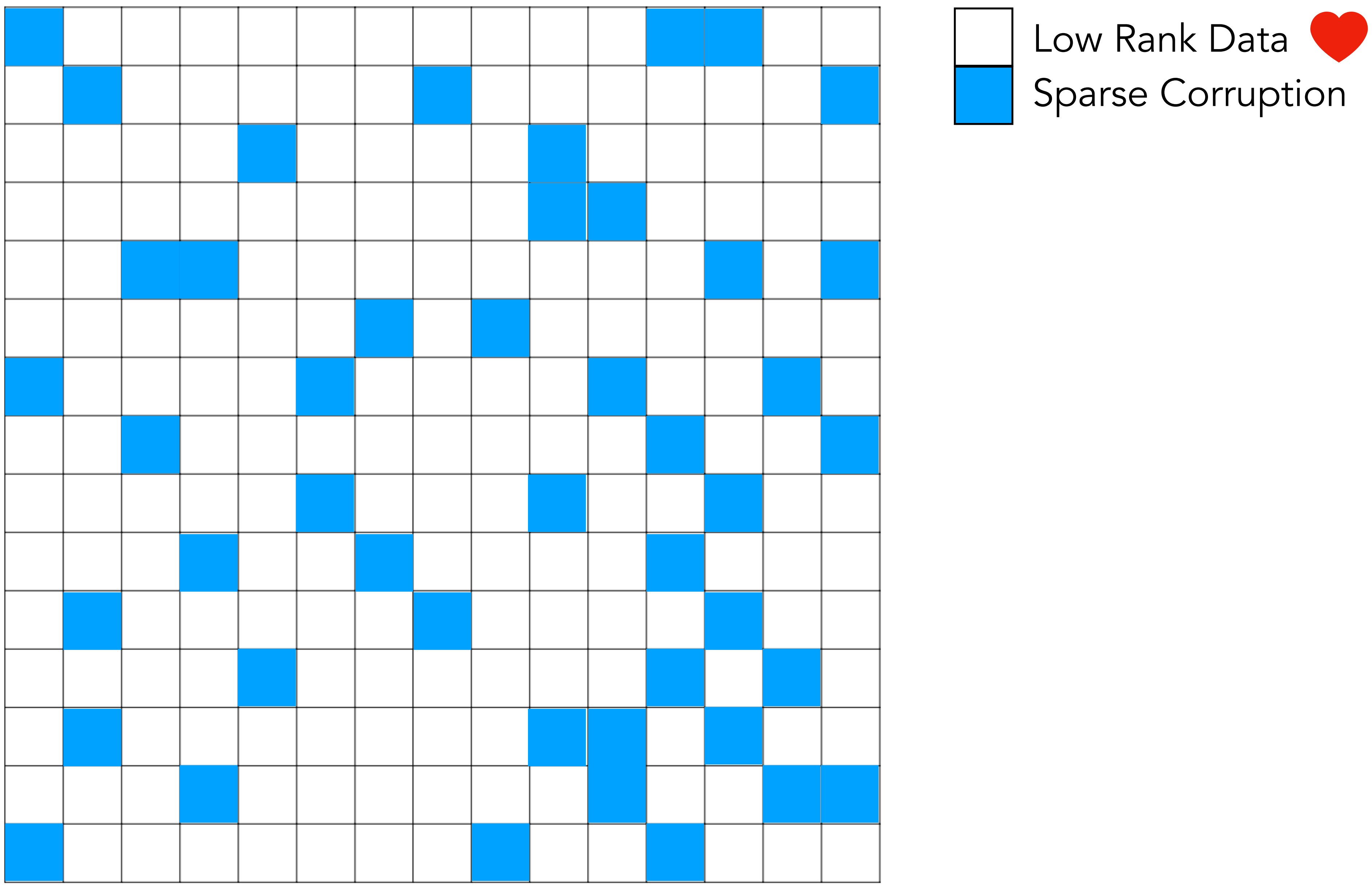

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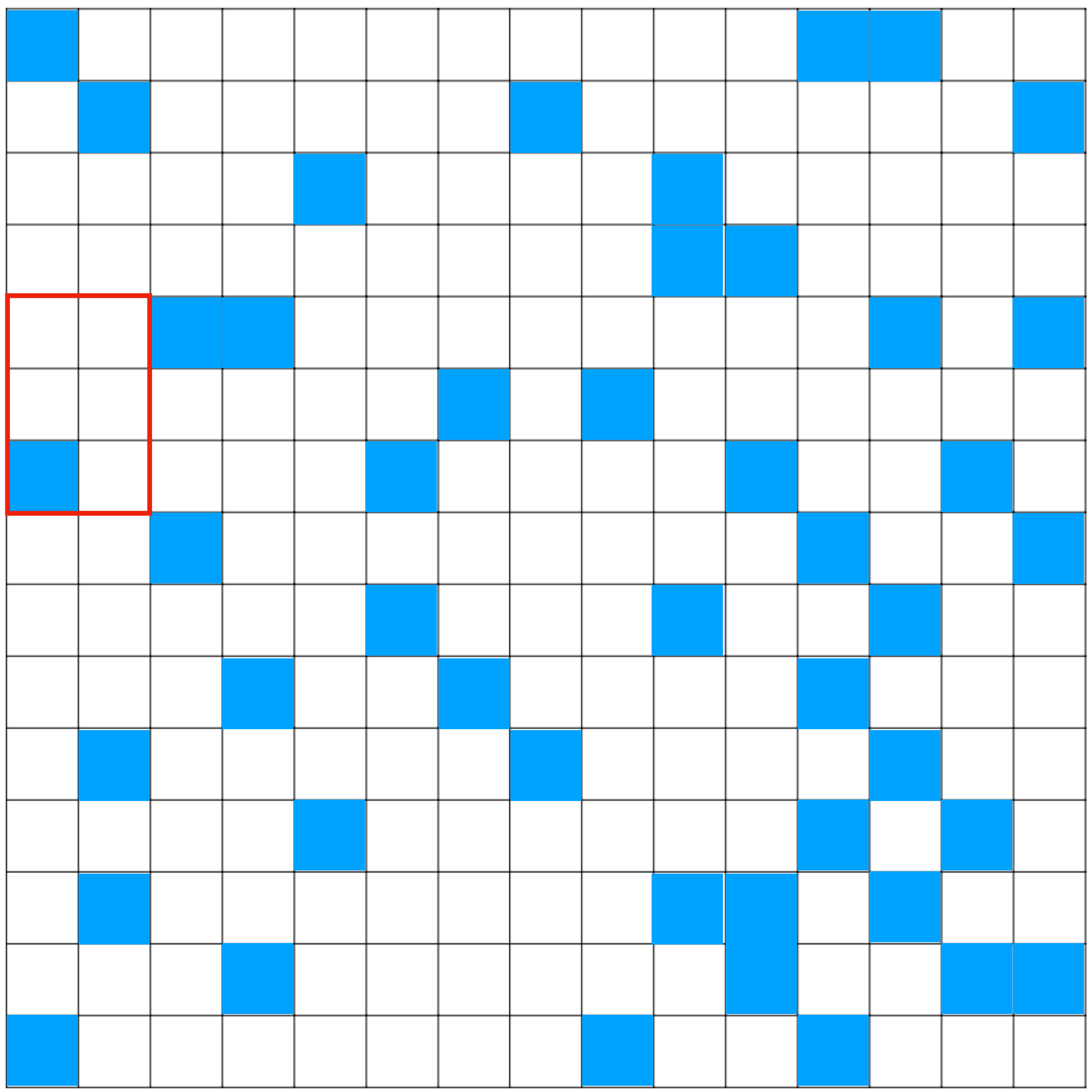

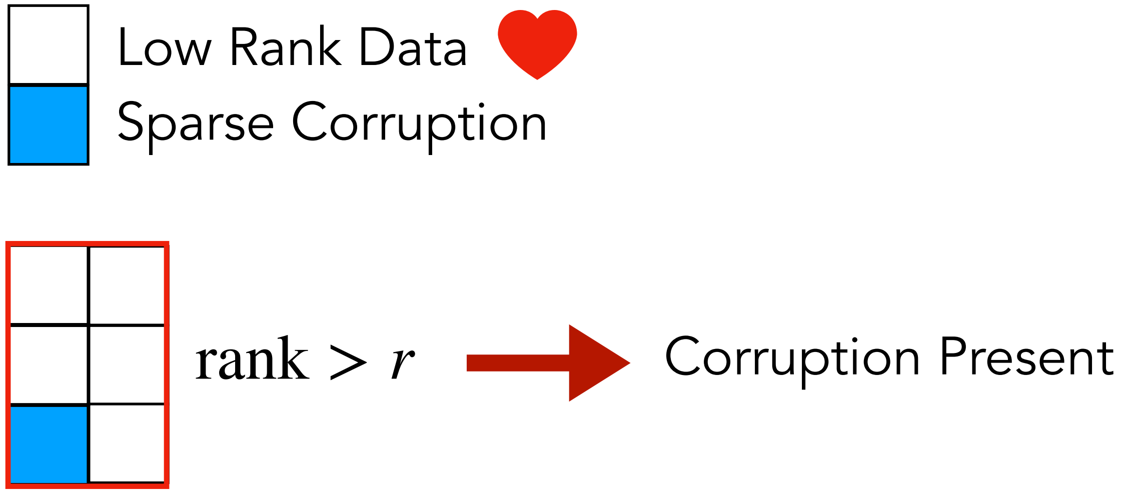

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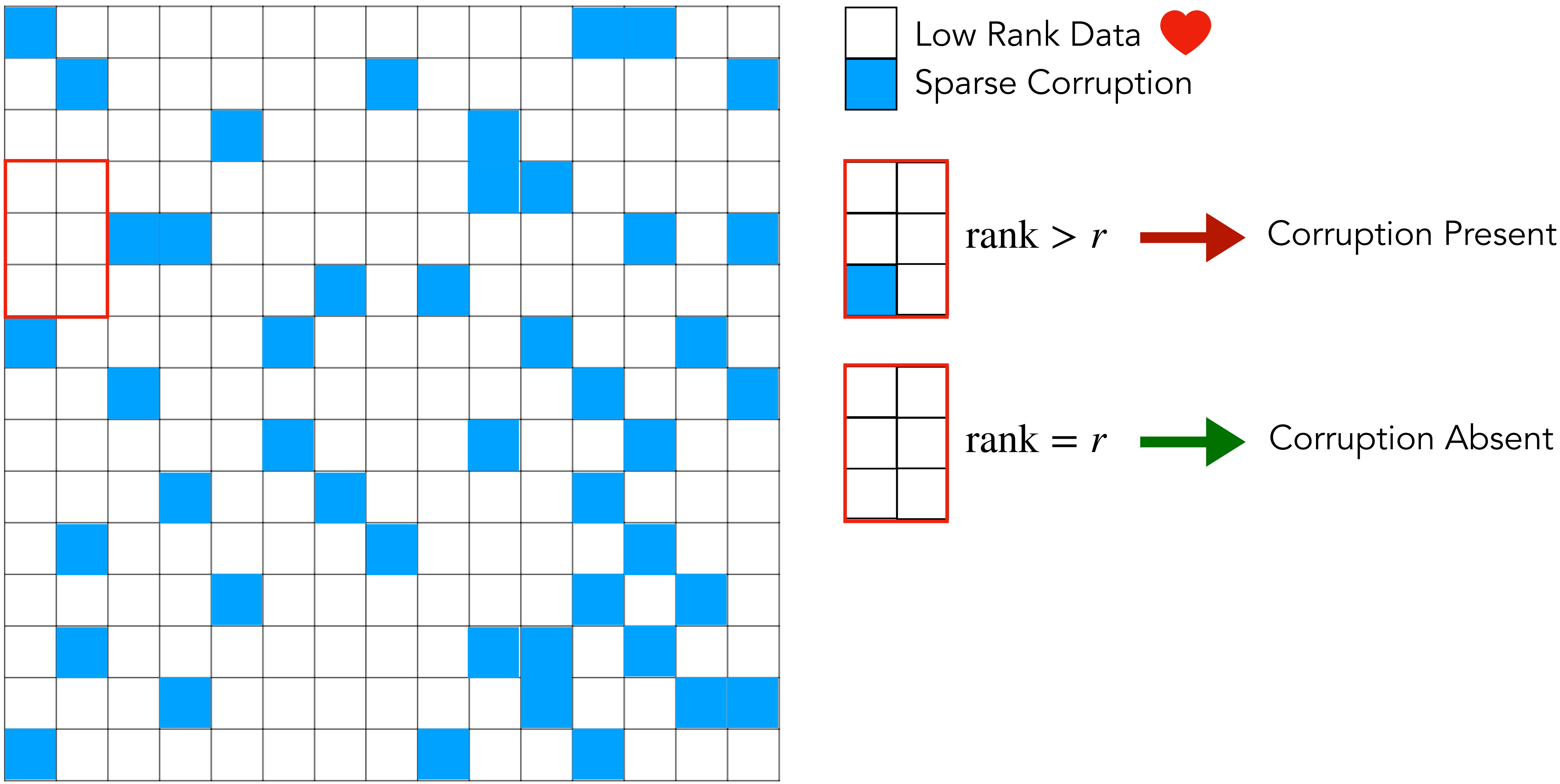

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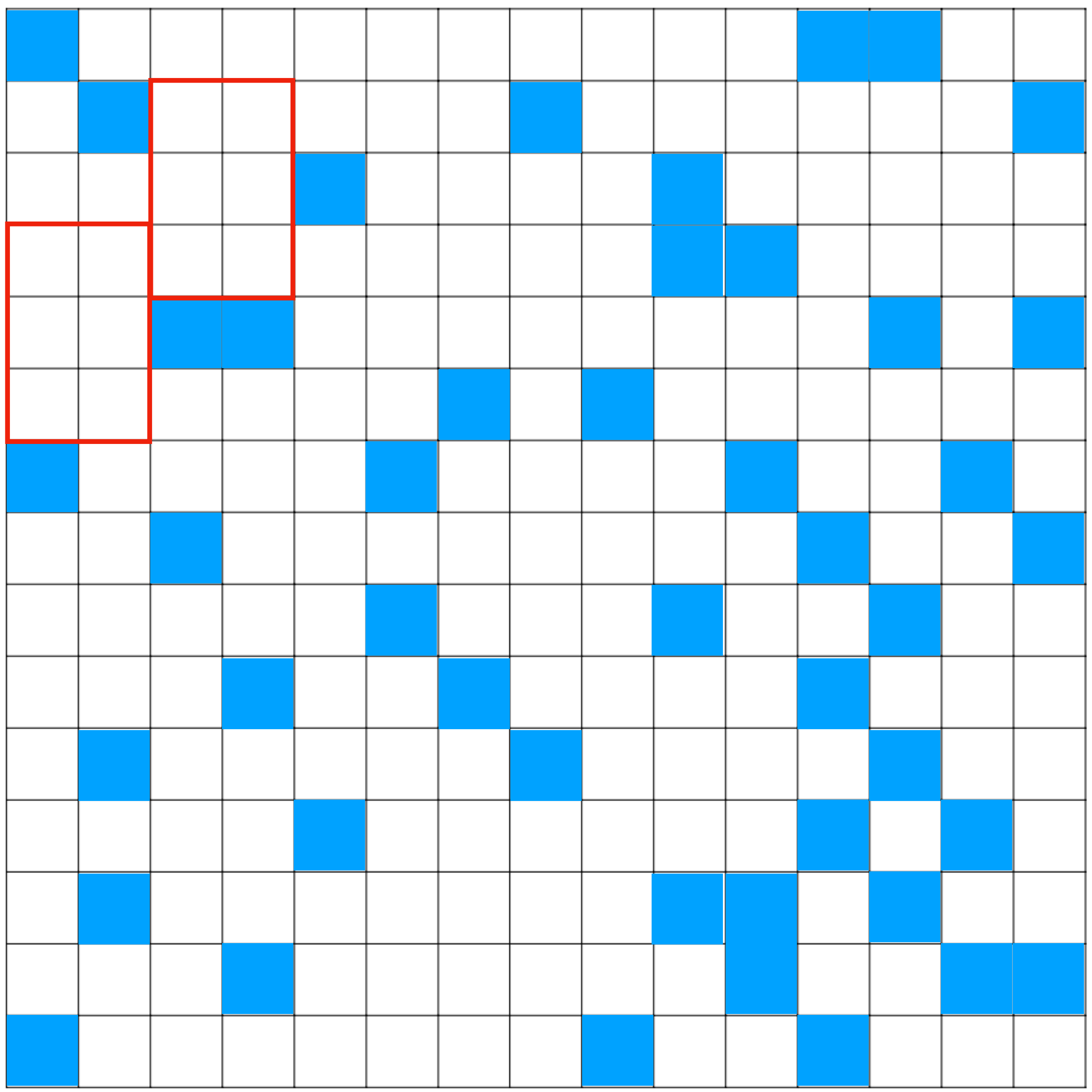
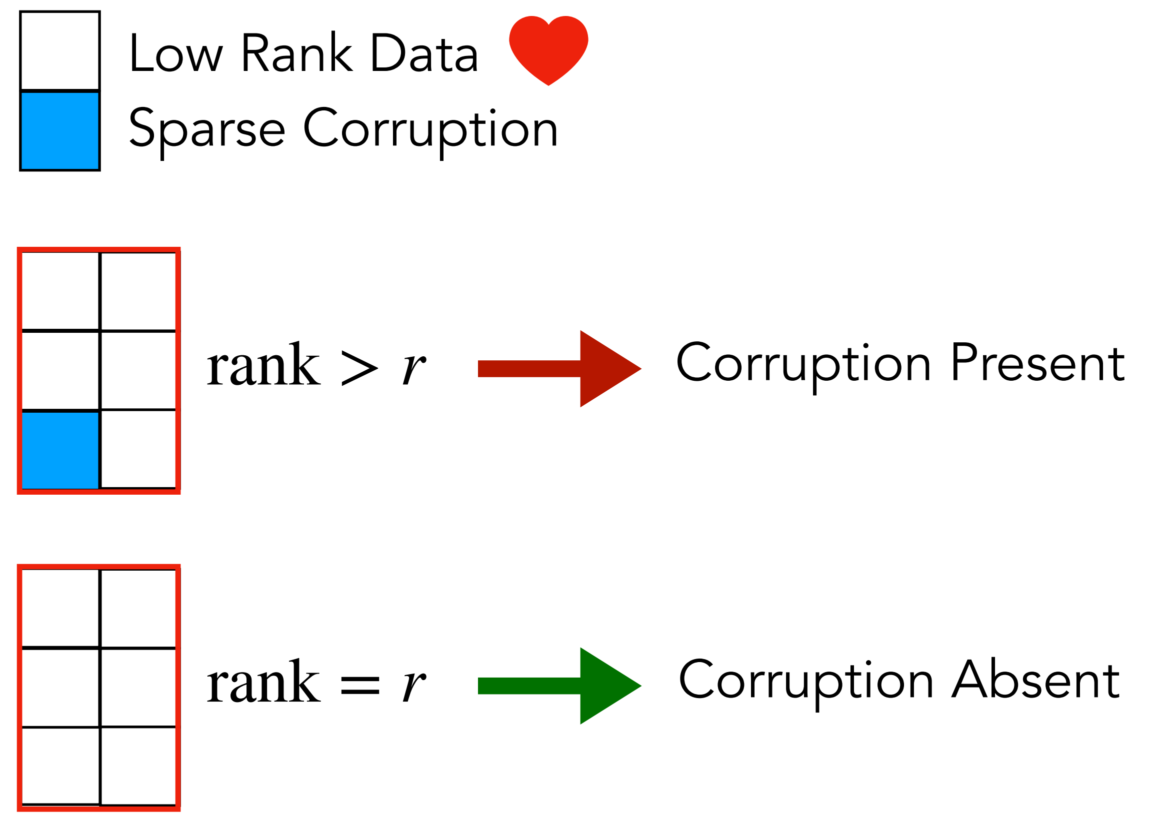
D. Pimentel-Alarcón and R. Nowak, "Random consensus robust pca," *Electronic Journal of Statistics*, vol. 11, no. 2, pp. 5232–5253, 2017.

Applications - RPCA Algorithms





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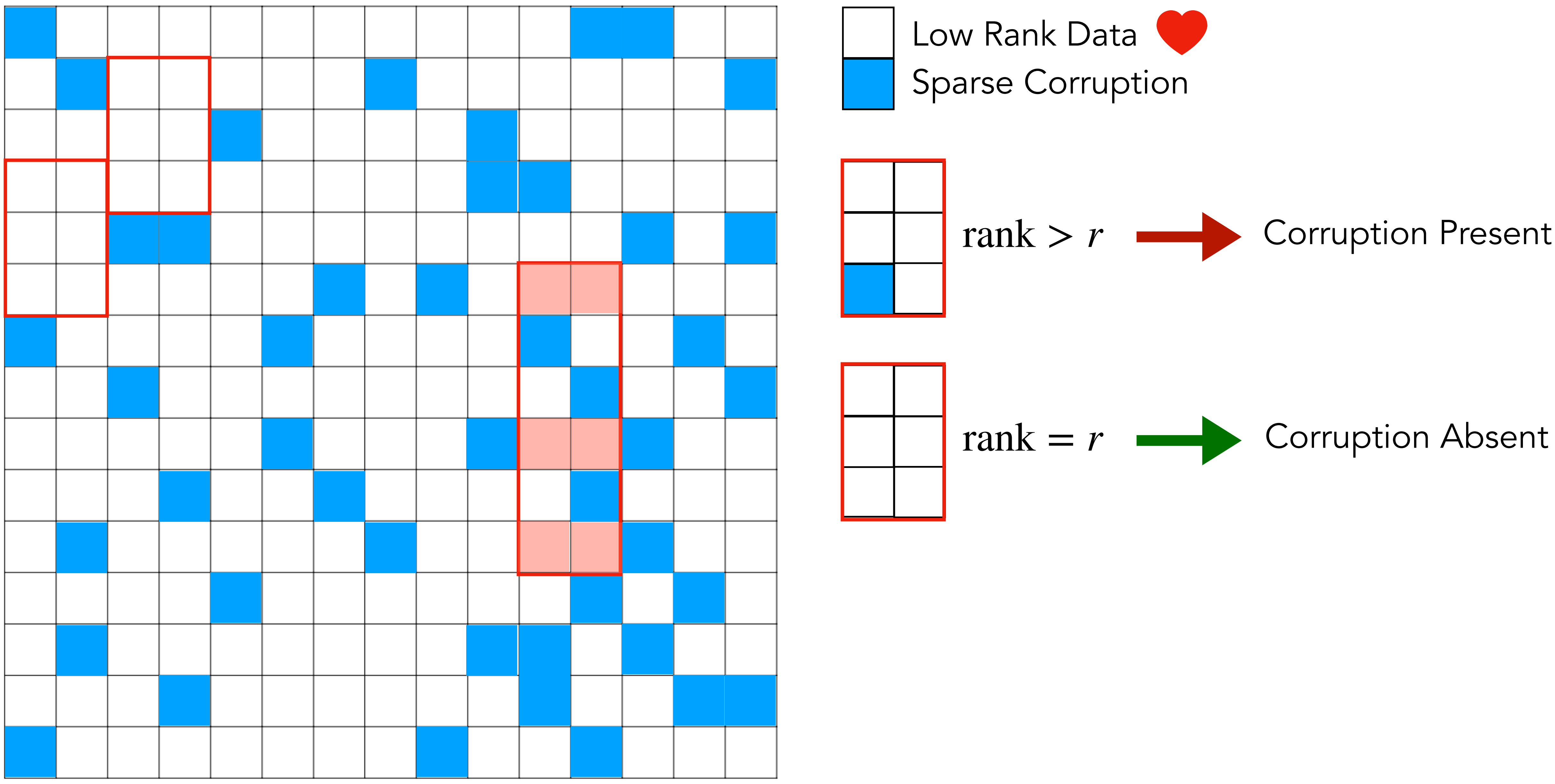

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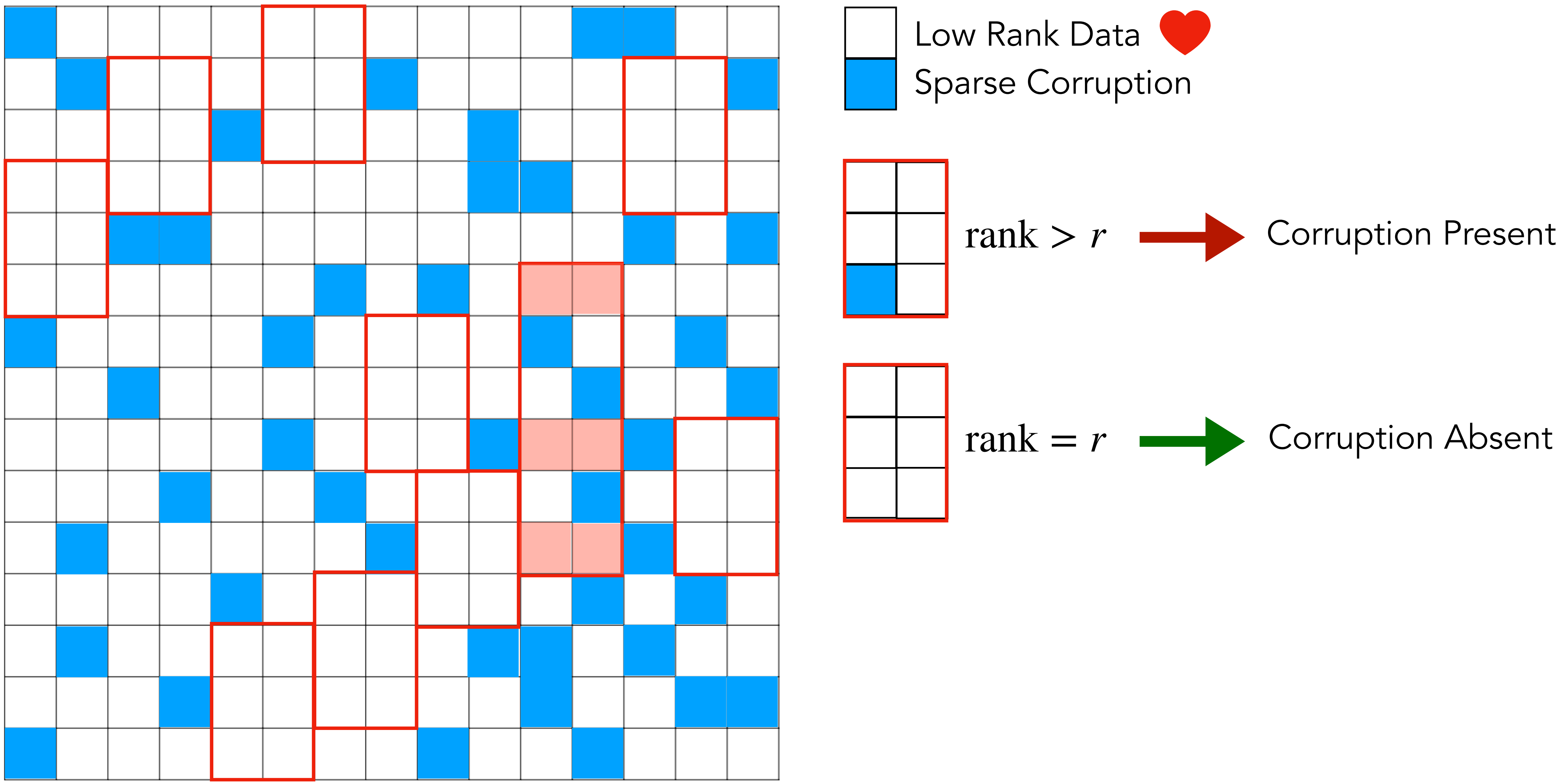

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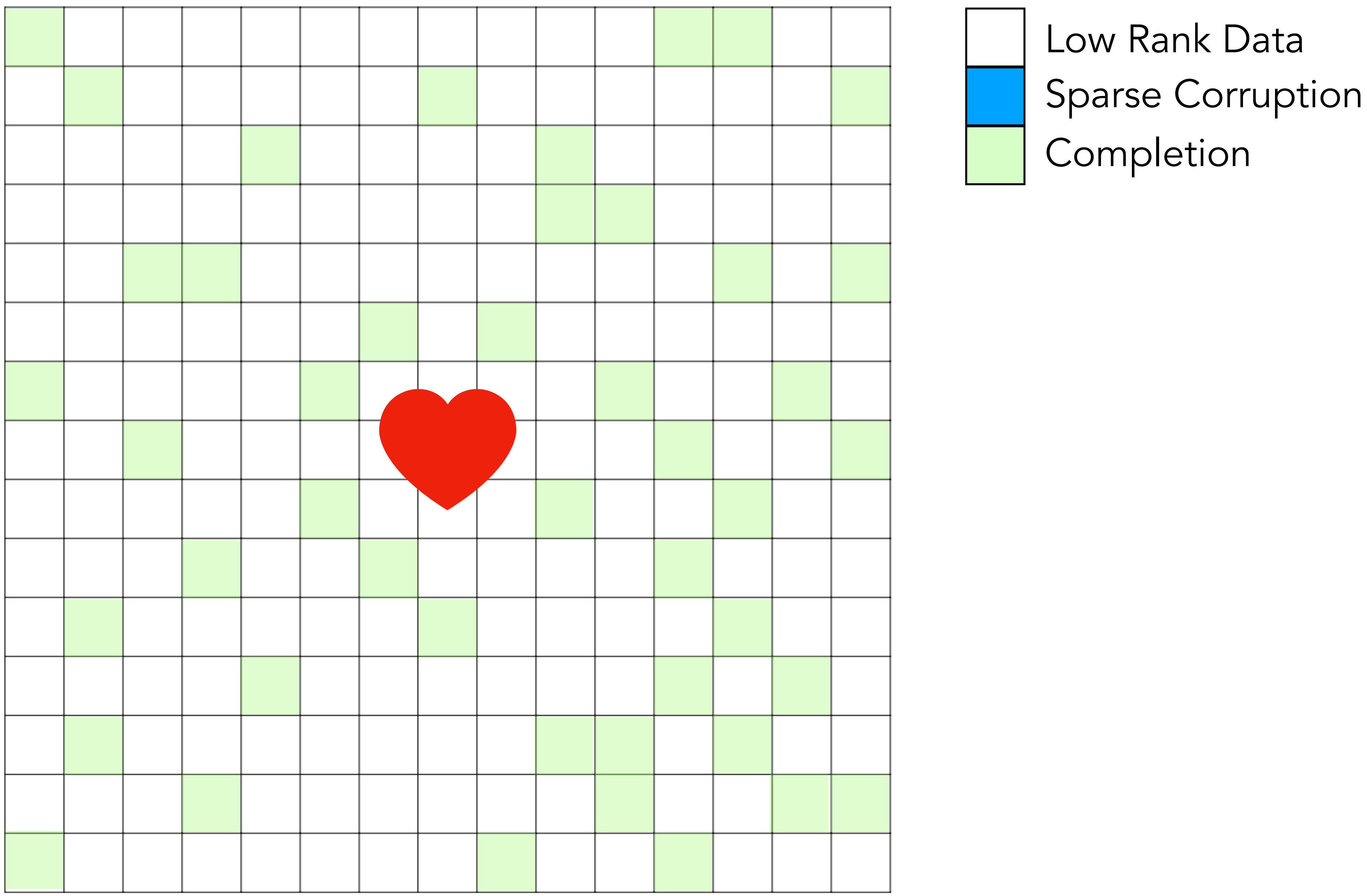

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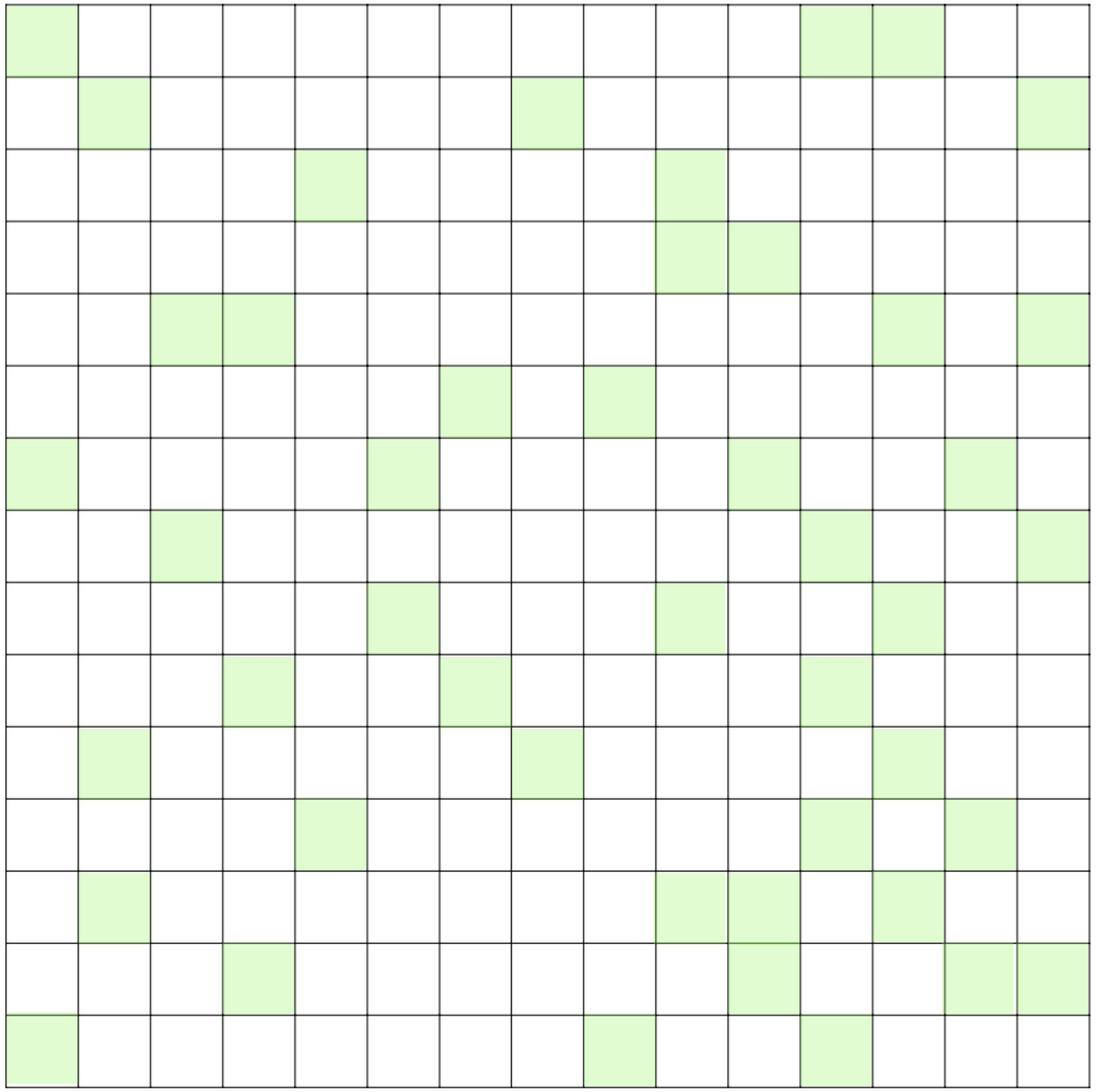

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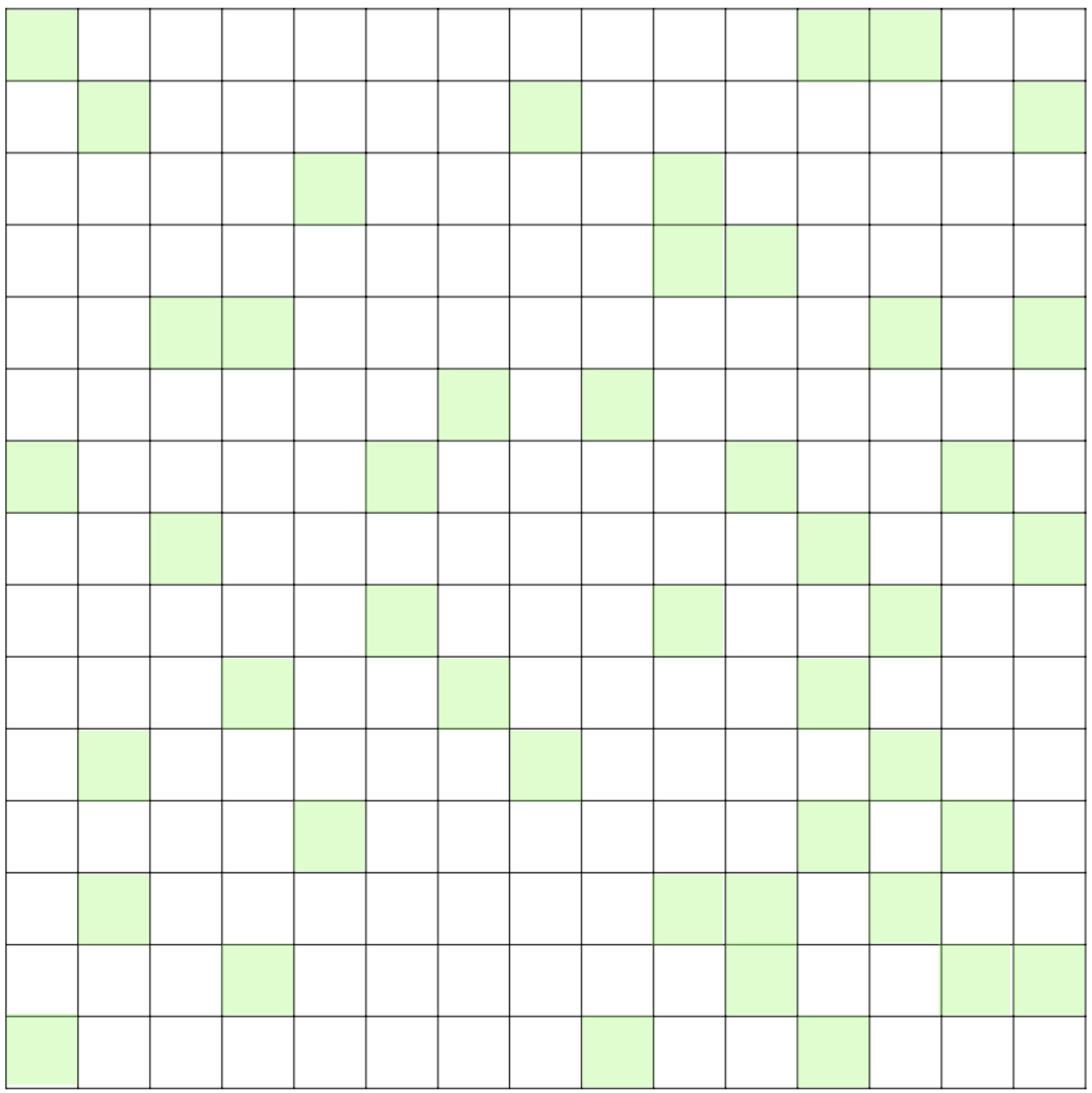
Standard RPCA Methods:

- Use Optimization approaches
- Require coherence / incoherence in data
- Work with high probability

Applications - RPCA Algorithms



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Subspace Reconstruction Based Algorithm:

- Algebraic / Geometric approach
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- No assumptions on the data
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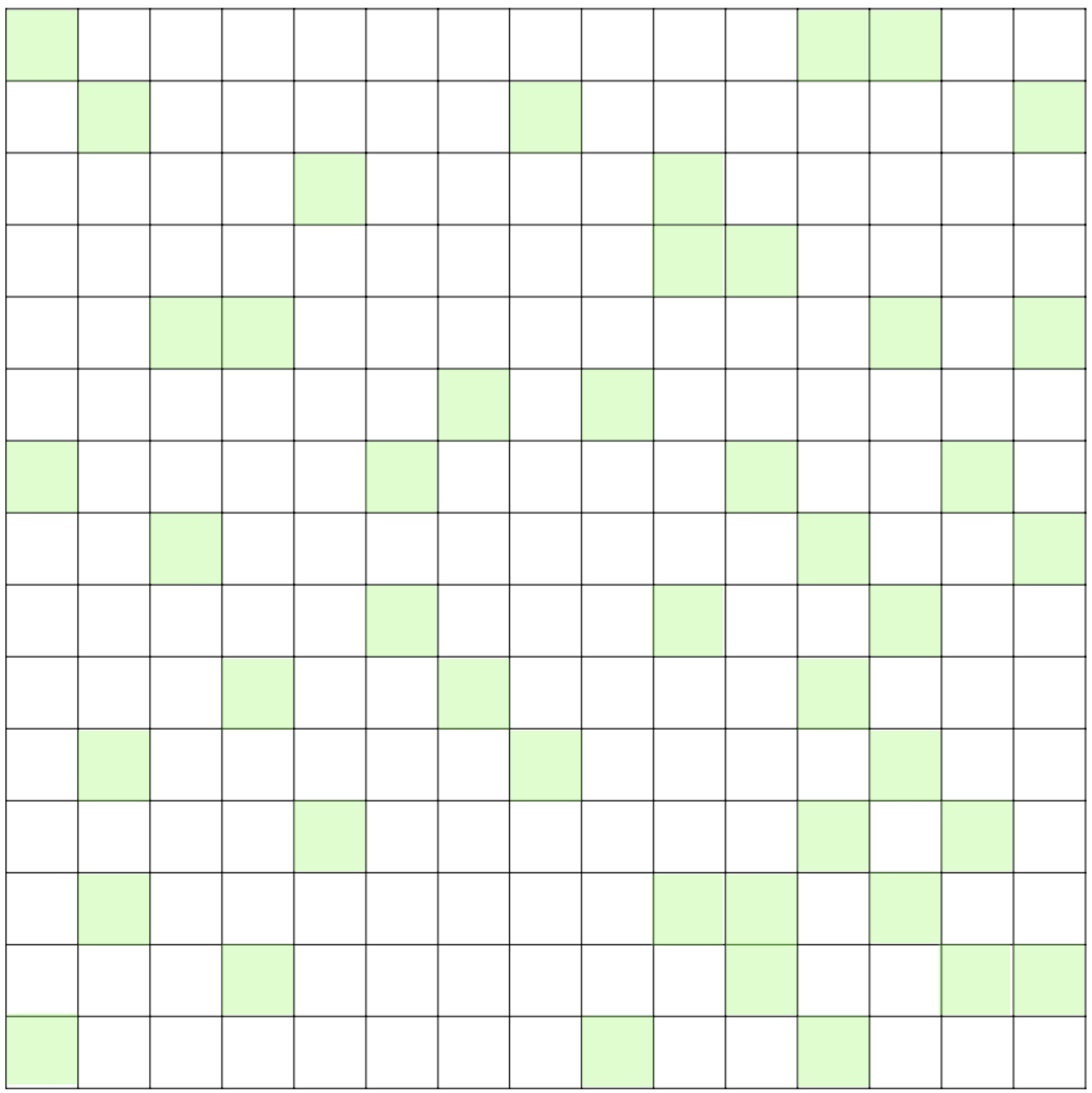
:)

Applications - RPCA Algorithms



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Standard RPCA Methods:

- Use Optimization approaches
- Require coherence / incoherence in data
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- HAVE NOISY BOUNDS

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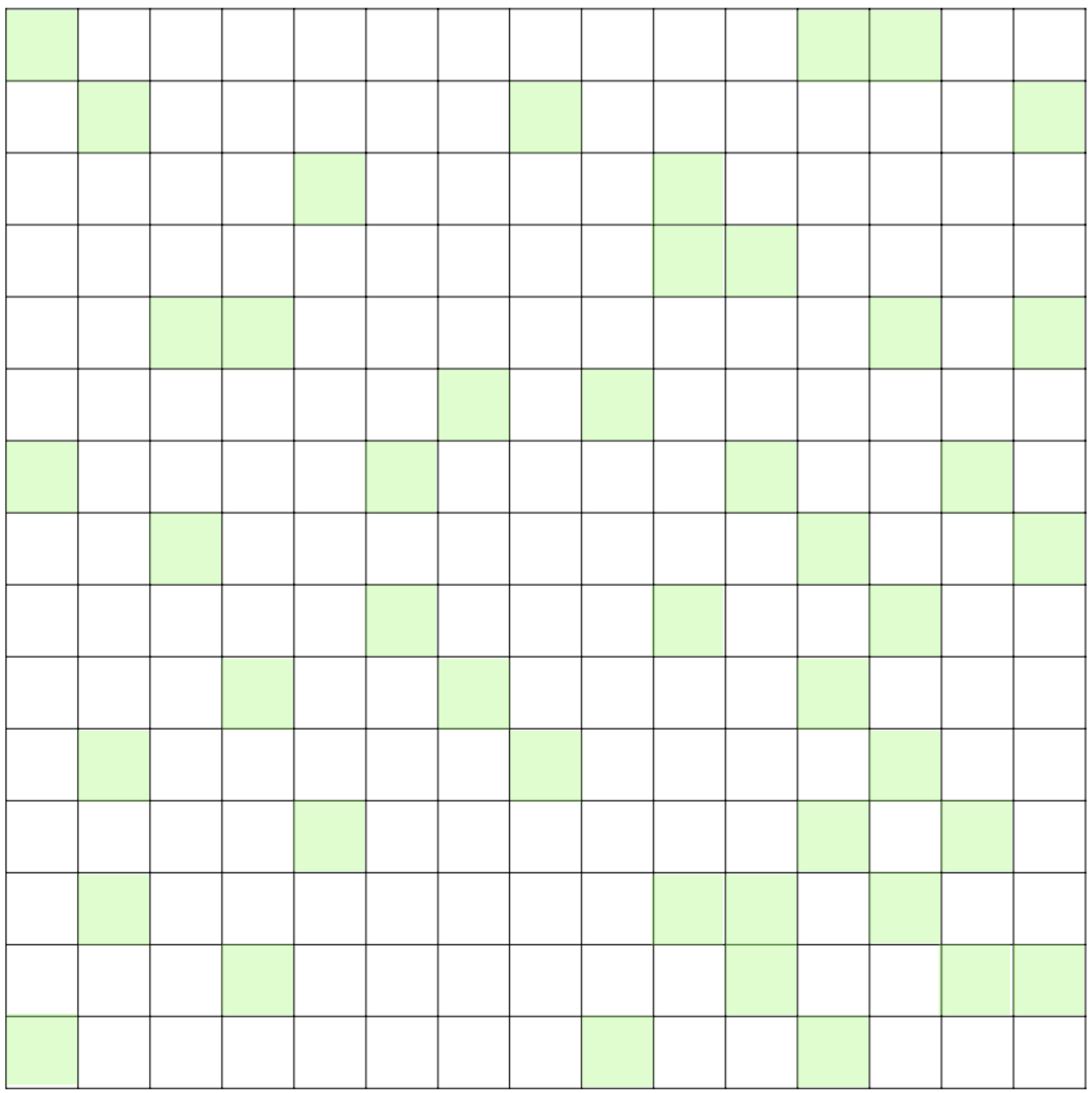
Applications - RPCA Algorithms



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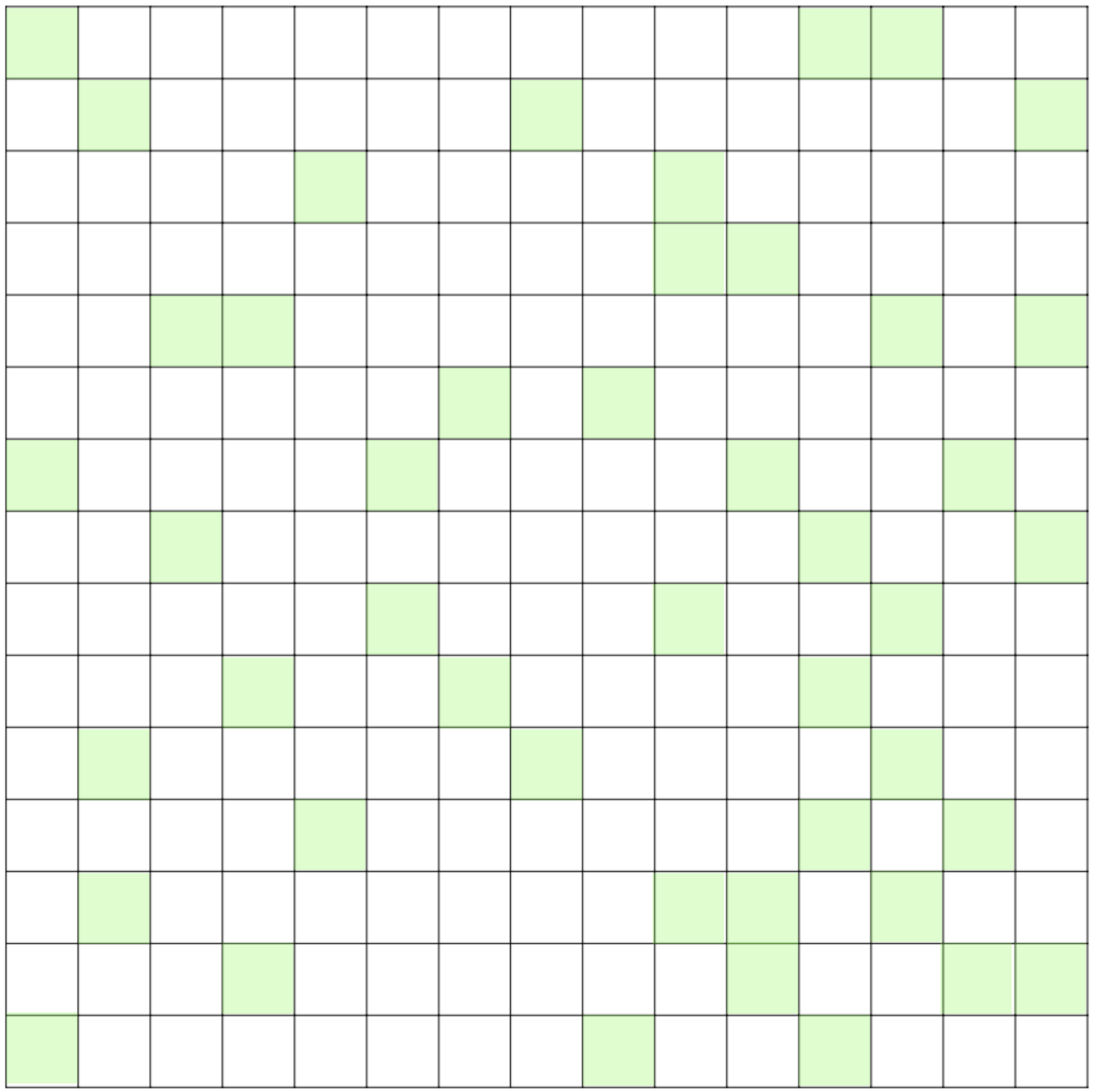
Applications - RPCA



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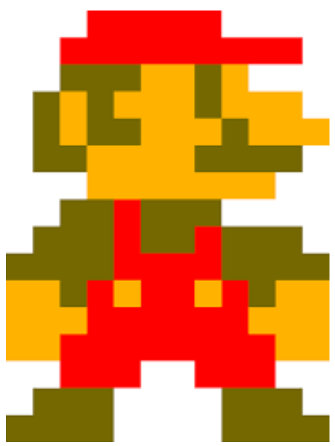
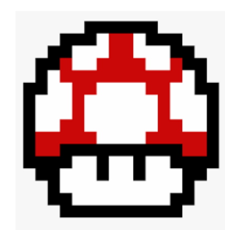
:)

Applications - Our Paper



Existing Subspace-Reconstruction Based LRMC Theory, RPCA Algorithms, etc (Noiseless)

+



Understanding of Noisy Subspace Reconstruction (Noisy Bound)

=

Generalization of existing results to Noisy Cases

(Or at least a step in this direction)

Outline

1. Problem Setup - Subspace Estimation
2. Motivation - Missing Data
3. Previous Work: Noiseless case
4. This Paper - Noisy Data and Estimation Bound
5. Applications
6. Conclusions

Conclusions

In our work, we have

- Generalized subspace estimation method to deal with noisy data

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In our work, we have

- Generalized subspace estimation method to deal with noisy data
- Bounded the error in approximating the optimal subspace estimator using the deterministic conditions for subspace identifiability
- Experimentally verified that sampling patterns affect the construction and bound

References

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All code for experiments can be found at github.io/ksrivastava1